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The Riemann-Hilbert Problem and Soliton Solution of a Variable Coefficient Nonlocal GI Equation

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Abstract: This paper adopts the Riemann-Hilbert method to investigate a variable coefficient nonlocal Gerdjikov-Ivanov (GI) equation. A solvable regular Riemann-Hilbert problem with simple zeros is constructed through spectral analysis. Furthermore, the paper provides the determinant form of N-solutions for the equation.

Keywords: Riemann-Hilbert method; N-solutions; Variable coefficient nonlocal Gerdjikov-Ivanov (GI) equation; Spectral analysis

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1. Introduction

The Riemann-Hilbert (RH) method, based on inverse scattering theory and considered a modern version of inverse scattering methods, offers an efficient solution approach. RH methods find wide applications for solving soliton solutions and analyzing the long-term behavior of integrable systems [1–4]. RH method also plays a vital role in diverse fields like infinite-dimensional Grassmann manifolds, quantum fields, statistical mechanics models, holomorphic vector bundles, combinatorial mechanics, orthogonal polynomials, and random matrix theories [5–6].

This paper employs the RH method to solve the variable coefficient nonlocal GI equation and successfully provides a determinant representation for N-soliton solutions. Section 2 introduces a class of variable coefficient nonlocal GI equations and validates their Lax pairs. Section 3 conducts spectral analysis on Lax pairs to establish the RH problem for variable coefficient nonlocal GI equations under specific boundary conditions. In Section 4, the paper derives a solvable regularized RH problem through transformations. Finally, the study obtain a series of soliton solutions for nonlocal GI equations and provides visual representations of these solitons.

2. Variable coefficient nonlocal GI equation

In reference, the paper first proposed the GI equation in the following form [7]:

$$iu_t + u_{xx} - iu^2 u_x^* + \frac{1}{2} u^3 u^{*2} = 0 \tag{1}$$

Equation 1 is a third-order derivative nonlinear Schrödinger equation, also referred to as the DNLS III equation.

The nonlocal form of the GI equation is presented in **Equation 2** [8].

$$iu_t(x,t) + u_{xx}(x,t) + iu^2(x,t)u_x(-x,-t) + \frac{1}{2}u^3(x,t)u^2(-x,-t) = 0 \tag{2}$$

This paper studies the variable coefficient nonlocal GI equation, which has the following form.

$$iu_t(x,t) + \delta(t)u_{xx}(x,t) + i\delta(t)u^2(x,t)u_x(-x,-t) + \frac{1}{2}\delta(t)u^3(x,t)u^2(-x,-t) = 0 \tag{3}$$

The Lax pair is given by **Equation 4**.

$$\begin{cases} \Psi_x + i\lambda^2\sigma_3\Psi = \widehat{U}\Psi, \\ \Psi_t + 2i\lambda^4\delta(t)\sigma_3\Psi = \widehat{V}\Psi \end{cases} \tag{4}$$

Where

$$\widehat{U} = \lambda Q - \frac{1}{2}iQ^2\sigma_3$$

$$\widehat{V} = \delta(t)(2\lambda^3 Q + i\lambda^2 Q^2\sigma_3 + i\lambda\sigma_3 Q_x + \frac{1}{2}Q_x Q - \frac{1}{2}QQ_x + \frac{1}{4}iQ^4\sigma_3)$$

With

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Q = \begin{pmatrix} 0 & u(x,t) \\ u(-x,-t) & 0 \end{pmatrix}$$

3. Construction of the Riemann-Hilbert problem

Making a transformation,

$$\Phi = \Psi e^{i\lambda^2\sigma_3 x + 2i\lambda^4\sigma_3\delta(t)t} \tag{5}$$

then the Jost function $\Phi(x,t;\lambda)$ has the property

$$\Phi(x,t;\lambda) \sim I, |x| \to \infty \tag{6}$$

and it satisfies the Lax pair

$$\begin{cases} \Phi_x + i\lambda^2[\sigma_3, \Phi] = \widehat{U}\Phi, \\ \Phi_t + 2i\lambda^4\delta(t)[\sigma_3, \Phi] = \widehat{V}\Phi \end{cases} \tag{7}$$

where $[\sigma_3, \Phi] = \sigma_3\Phi - \Phi\sigma_3$

The Lax pair in Equation 7 can also be expressed in differential form as follows:

$$d(e^{i\theta(x,t;\lambda)\hat{\sigma}_3}\Phi) = e^{i\theta(x,t;\lambda)\hat{\sigma}_3}(M\Phi) \tag{8}$$

With $M = \widehat{U}dx + \widehat{V}dt, \theta(x,t;\lambda) = \lambda^2 x + 2\lambda^4 \delta(t)t$

Integrating Equation 8 from $(\mp\infty, t)$ to (x, t) to get

$$\Phi_{\pm} = I \pm \int_{\mp\infty}^{x} e^{-i\lambda^2(x-\zeta)\widehat{\sigma}_3}\, \widehat{U}(\zeta,t;\lambda)\Phi(\zeta,t;\lambda)d\zeta \tag{9}$$

where $e^{\widehat{\sigma}_3}A = e^{\sigma_3}Ae^{-\sigma_3}$

$[\Phi_+]_k$ denotes the k-th column vector, and k = 1,2. It can be seen that $[\Phi_-]_1, [\Phi_+]_2$ are analytic for $\lambda \in \mathbb{C}_+$ and $[\Phi_+]_1, [\Phi_-]_2$ are analytic for $\lambda \in \mathbb{C}_-$ Here

$$\mathbb{C}_+ = \left\{\lambda \,|\, arg\,\lambda \in \left(0,\frac{\pi}{2}\right) \cup \left(\pi,\frac{3\pi}{2}\right)\right\} \qquad \mathbb{C}_- = \left\{\lambda \,|\, arg\,\lambda \in \left(\frac{\pi}{2},\pi\right) \cup \left(\frac{3\pi}{2},2\pi\right)\right\}$$

The study defines $\Lambda = e^{-i\lambda^2\sigma_3 x}$, it is easy to see that $\Phi_-\Lambda$ and $\Phi_+\Lambda$ are two different solutions of the Lax pair in **Equation 4**, and they satisfy

$$\Phi_-\Lambda = \Phi_+\Lambda S(\lambda), \lambda \in R \cup iR \tag{10}$$

where the scattering matrix S(λ) does not depend on x and t.

From **Equation 10**, the derived scattering matrix S(λ) satisfies

$$S(\lambda) = \lim_{x\to+\infty} \Lambda^{-1}\Phi_-\Lambda = I + \int_{-\infty}^{+\infty} e^{i\lambda^2 x'\widehat{\sigma}_3}\, \widehat{U}(x',t;\lambda)\Phi_- dx', \lambda \in R \cup iR \tag{11}$$

According to the analytic property of Φ_-, the study can deduce that S_{11} is an analytic extension in C_+ and S_{22} is analytic in C_-. A new Jost solution P^+ can be defined as

$$P^+ = ([\Phi_-]_1[\Phi_+]_2) = \Phi_- H_1 + \Phi_+ H_2 = \Phi_+\Lambda \begin{pmatrix} s_{11} & 0 \\ s_{21} & 1 \end{pmatrix} \Lambda^{-1} \tag{12}$$

P^+ is analytic in $\mathbb{C}_+$ with the asymptotic behavior.

$$P^+(x,t;\lambda) \to I, \quad \lambda \in C_+ \to \infty \tag{13}$$

Similarly, P^- exhibits the following form:

$$P^- = \begin{pmatrix} [\Phi_-^{-1}]_1 \\ [\Phi_+^{-1}]_2 \end{pmatrix} = H_1\Phi_-^{-1} + H_2\Phi_+^{-1} = \Lambda \begin{pmatrix} s_{22} & -s_{12} \\ 0 & 1 \end{pmatrix} \Lambda^{-1}\Phi_+^{-1} \tag{14}$$

P^- is analytic in $\mathbb{C}_-$ with the following asymptotic behavior:

$$P^-(x,t;\lambda) \to I, \lambda \in C_- \to \infty \tag{15}$$

The above analysis reveals that P^+ and P^- are analytics in $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively. Therefore, an RH problem can be constructed as follows

$$P^-P^+ = \Omega(x,t;\lambda) = \Lambda \begin{pmatrix} 1 & -s_{12} \\ s_{21} & 1 \end{pmatrix} \Lambda^{-1} \quad , \lambda \in R \cup iR \tag{16}$$

Next, the study analyzes the scattering data in Equation 10 and obtain

$$S_t = \lim_{x\to\infty} \Lambda^{-1}\Phi_{-,t}\Lambda \tag{17}$$

Inserting $\Phi_- = \Phi_+\Lambda S(\lambda)\Lambda^{-1}$ into Lax pair in **Equation 7** and considering $\widehat{V} \to 0$ as $x \to +\infty$, the study can derive

$$S_t + 2i\lambda^4\delta(t)[\sigma_3, S] = 0 \tag{18}$$

Expanding the **Equation 18**, the scattering matrix data is obtained as follows:

$$s_{11,t} = s_{22,t} = 0, s_{12,t} + 4i\lambda^4\delta(t) = s_{21,t} - 4i\lambda^4\delta(t) = 0$$

4. Regularization of the Riemann-Hilbert problem

Defining the operator L

$$LA = L\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{22}^* & -a_{21}^* \\ -a_{12}^* & a_{11}^* \end{pmatrix} \triangleq A^\dagger \tag{19}$$

The study can verify that

$$\Phi^\dagger(x, \lambda^*) = \Phi^{-1}(x, \lambda), \lambda \in R \cup iR \tag{20}$$

And

$$(P^+)^\dagger(\lambda^*) = P^-(\lambda), \lambda \in R \cup iR \tag{21}$$

In the previous section, the study proved that S_{11} is independent of time. It is obvious that $S_{11}(\lambda)$ is an odd function. For every zero point λ_k, there must exist another corresponding zero point $-\lambda_k$, where k=1,,2,...N Assuming all zero points are simple, then $(P^+(\pm\lambda_k))$ and $ker(P^-(\pm\lambda_k^*))$ are one-dimensional spaces produced by $|v_k\rangle$ and $\langle v_k|$. The $|v_k\rangle$ and $\langle v_k|$ are single-column vectors and row vectors, respectively. Here $|v_k\rangle = \langle v_k|^\dagger$.

Obviously,

$$P^+(\lambda_k)|v_k\rangle = 0, \quad \langle v_k|P^-(\lambda_k^*) = 0 \tag{22}$$

To obtain the regular RH problem, a matrix function is constructed

$$\Upsilon_k(\lambda) = I + \frac{A_k}{\lambda - \lambda_k^*} - \frac{\sigma_3 A_k \sigma_3}{\lambda + \lambda_k^*} \tag{23}$$

where $\Upsilon_k(\lambda)$ is a meromorphic function with two simple poles $\lambda = \pm\lambda_k^*$ in $\mathbb{C}_-$.

After direct computation, the following is obtained

$$\Upsilon_k^{-1}(\lambda) = I + \frac{A^\dagger}{\lambda - \lambda_k} - \frac{\sigma_3 A^\dagger \sigma_3}{\lambda + \lambda_k} \tag{24}$$

Moreover, the following is obtained

$$det\left(P^+(\lambda)\Upsilon_k^{-1}(\lambda) \neq 0\right), \lambda = \pm\lambda_k; det\left(\Upsilon_k(\lambda)P^-(\lambda)\right) \neq 0, \lambda = \pm\lambda_k^*$$

The $\Upsilon(\lambda)$ and $\Upsilon^{-1}(\lambda)$ have the following expansion:

$$\Upsilon(\lambda) = \prod_{k=1}^N \Upsilon_k(\lambda), \Upsilon^{-1}(\lambda) = \prod_{k=1}^N \Upsilon_k^{-1}(\lambda) \tag{25}$$

The study defines

$$P_+(\lambda) = P^+(\lambda)\Upsilon^{-1}(\lambda), \quad P_-(\lambda) = \Upsilon(\lambda)P^-(\lambda) \tag{26}$$

then, a regular RH problem can be constructed as follows:

$$P_-(\lambda)P_+(\lambda) = \Upsilon(\lambda)G(\lambda)\Upsilon^{-1}(\lambda), \lambda \in R \cup iR \tag{27}$$

Therefore, noting the canonical normalization conditions in **Equations 13** and **15**, the study can derive the solution of a non-regular RH problem in **Equation 16** as

$$P^{+}(\lambda)= P_{+}(\lambda)\Upsilon(\lambda), \quad P^{-}(\lambda) = \Upsilon^{-1}(\lambda)P_{-}(\lambda) \tag{28}$$

Where only the simple zeros problem is considered.

According to **Equation 22** and the analytic property of P_{+}, the study has $\Upsilon(\lambda_k)|v_k\rangle = 0$ and $\langle v_k|\Upsilon^{-1}(\lambda_k) = 0$, indicating that A_k is linearly related to $\langle v_k|$. Through direct computation, the following is obtained

$$A_k = \frac{\lambda_k-\lambda_k^*}{2}\begin{pmatrix}\alpha_k & 0\\ 0 & \alpha_k^*\end{pmatrix}|w_k\rangle\langle w_k|, \alpha_k^{-1} = \langle w_k|\begin{pmatrix}\lambda_k^* & 0\\ 0 & \lambda_k\end{pmatrix}|w_k\rangle \tag{29}$$

Where $|w_k\rangle = \Upsilon_{k-1}(\lambda_k)\cdots\Upsilon_1(\lambda_k)|v_k\rangle$.

5. The soliton solutions

Based on the analysis above, the study is prepared to determine the soliton solutions for **Equation 3**. According to the expansion of $\Upsilon_k^{-1}(\lambda)$ and A_k there exists a series of vectors $|l_k\rangle$ such that

$$Res_{\lambda=\lambda_k}\Upsilon^{-1}(\lambda) = \lim_{\lambda\to\lambda_k}(\lambda-\lambda_k)\Upsilon^{-1}(\lambda) = |v_k\rangle\langle l_k| \tag{30}$$

where $\langle l_k| = |l_k\rangle^{\dagger}$.

Considering the symmetric property

$$\sigma_3\Upsilon_k(\lambda)\sigma_3 = \Upsilon_k(-\lambda) \tag{31}$$

The study can derive the compact form of $\Upsilon(\lambda)$ and $\Upsilon^{-1}(\lambda)$ as follows:

$$\Upsilon(\lambda) = I + \sum_{k=1}^{N}\left(\frac{B_k}{\lambda-\lambda_k^*} - \frac{\sigma_3 B_k\sigma_3}{\lambda+\lambda_k^*}\right), \Upsilon^{-1}(\lambda) = I + \sum_{k=1}^{N}\left(\frac{B_k^{\dagger}}{\lambda-\lambda_k} - \frac{\sigma_3 B_k^{\dagger}\sigma_3}{\lambda+\lambda_k}\right) \tag{32}$$

where $B_k = |l_k\rangle\langle v_k|$.

According to the identity $\Upsilon(\lambda)\Upsilon^{-1}(\lambda) = \Upsilon^{-1}(\lambda)\Upsilon(\lambda) = I$, the study has

$$Res_{\lambda=\lambda_k}\Upsilon(\lambda)\Upsilon^{-1}(\lambda) \tag{33}$$

The study can derive

$$\Upsilon(\lambda_k)B_k^{\dagger} = 0 \tag{34}$$

which indicates

$$\left[I + \sum_{k=1}^{N}\frac{|l_k\rangle\langle v_k|}{\lambda_j-\lambda_k^*} - \frac{\sigma_3|l_k\rangle\langle v_k|\sigma_3}{\lambda_j+\lambda_k^*}\right]|v_j\rangle = 0, j = 1,2\cdots N \tag{35}$$

And

$$|v_j\rangle = \sum_{k=1}^{N}\left(\frac{\sigma_3|l_k\rangle\langle v_k|\sigma_3|v_j\rangle}{\lambda_j-\lambda_k^*} - \frac{|l_k\rangle\langle v_k||v_j\rangle}{\lambda_j-\lambda_k^*}\right), j = 1,2\cdots N \tag{36}$$

Solving **Equations 35** and **36**, the following is obtained

$$|l_m\rangle_1 = \sum_{j=1}^{N} M^{-1}(m,j)|v_j\rangle_1, l_m\rangle_2 = \sum_{j=1}^{N}\widehat{M}^{-1}(m,j)|v_j\rangle_2 \tag{37}$$

The elements in M and $\widehat{M}$ present the following forms:

$$M_{jk}=\frac{\langle v_k|\sigma_3|v_j\rangle}{\lambda_j+\lambda_k^*}-\frac{\langle v_k||v_j\rangle}{\lambda_j-\lambda_k^*},\widehat{M}_{jk}=-\frac{\langle v_k|\sigma_3|v_j\rangle}{\lambda_j+\lambda_k^*}-\frac{\langle v_k||v_j\rangle}{\lambda_j-\lambda_k^*} \tag{38}$$

From Equation 16, the study has

$$(P^+)^{-1}-P^-=\widehat{\Omega}(P^+)^{-1}=(I-\Omega)(P^+)^{-1} \tag{39}$$

Based on the Plemelj formula, the RH problem in **Equation 27** has the following solution:

$$(P_+(\lambda))^{-1}=I+\frac{1}{2\pi i}\int_\Gamma\frac{\Upsilon(\eta)\widehat{\Omega}(\eta)\Upsilon^{-1}(\eta)(P_+(\eta))^{-1}}{\eta-\lambda}d\eta,\quad \lambda\in C_+ \tag{40}$$

where Γ is an arbitrary curve.

When $\lambda\to\ \infty$

$$P_+(\lambda)\to I+\frac{1}{2\pi i\lambda}\int_\Gamma\Upsilon(\eta)\widehat{\Omega}(\eta)\Upsilon^{-1}(\eta)(P_+(\eta))^{-1}d\eta \tag{41}$$

From **Equation 32**, the study has

$$\Upsilon(\lambda)=\ I+\frac{1}{\lambda}\sum_{k=1}^{N}(|l_k\rangle\langle v_k|-\sigma_3|l_k\rangle\langle v_k|\sigma_3),\lambda\to\ \infty \tag{42}$$

Expanding P^+ into a WKB form,

$$P^+=P_0^++\frac{P_1^+}{\lambda}+\frac{P_2^+}{\lambda^2}+O\left(\frac{1}{\lambda^3}\right) \tag{43}$$

The study inserts **Equation 43** into the Lax pair in **Equation 7** and contrasts the coefficients of various powers of λ, then obtain

$$[\sigma_3,P_0^+]=0,\quad i[\sigma_3,P_1^+]+Q\sigma_3P_0^+=0 \tag{44}$$

Where

$$P_1^+=\sum_{k=1}^{N}(|l_k\rangle\langle v_k|-\sigma_3|l_k\rangle\langle v_k|\sigma_3)+\frac{1}{2\pi i}\int_\Gamma\Upsilon(\xi)\widehat{\Omega}(\xi)\Upsilon^{-1}(\xi)(P_+(\xi))^{-1}d\xi \tag{45}$$

Equation 44 leads to

$$Q=i\sigma_3[\sigma_3,P_1^+] \tag{46}$$

Due to the Jump matrix $\Omega(\xi)\to I$ as $\lambda\to\infty$, the study can derive $\lim_{\lambda\to\infty}\widehat{\Omega}(\xi)=0$ and

$$Q=i\sigma_3\left[\sigma_3,\sum_{k=1}^{N}(|l_k\rangle\langle v_k|-\sigma_3|l_k\rangle\langle v_k|\sigma_3)\right] \tag{47}$$

which leads to the N-soliton solution of **Equation 3** as follows:

$$u=2i(P_1^+)_{12}=4i\sum_{k=1}^{N}(|l_k\rangle_1\langle v_k|_2)=-4i\frac{\det M_1}{\det M} \tag{48}$$

Where

$$M_1=\begin{pmatrix}M_{11} & \cdots & M_{1N} & |v_1\rangle_1\\ \vdots & & & \\ M_{N1} & \cdots & M_{NN} & |v_N\rangle_1\\ \langle v_1|_2 & \cdots & \langle v_N|_2 & 0\end{pmatrix} \tag{49}$$

With

$$|v_k\rangle = (c_k e^{-\theta_k} \quad e^{\theta_k})^T, M_{j,k} = \frac{-2\lambda_k^*}{\lambda_j^2 - \lambda_k^{*2}} c_j c_k^* e^{-\theta_j - \theta_k^*} - \frac{2\lambda_j}{\lambda_j^2 - \lambda_k^{*2}} e^{\theta_j + \theta_k^*}$$

According to **Equation 48**, when N = 1 the study has

$$det\,(M_1)_{2\times 2} = \begin{vmatrix} \frac{-2\lambda_1^*}{\lambda_1^2-\lambda_1^{*2}}|c_1|^2 e^{-\theta_1-\theta_1^*} - \frac{2\lambda_1}{\lambda_1^2-\lambda_1^{*2}} e^{\theta_1+\theta_1^*} & c_1 e^{-\theta_1} \\ e^{\theta_1^*} & 0 \end{vmatrix} = -c_1 e^{\theta_1^* - \theta_1},$$

$$det\,(M)_{1\times 1} = M_{11} = \frac{-2\lambda_1^*}{\lambda_1^2-\lambda_1^{*2}}|c_1|^2 e^{-\theta_1-\theta_1^*} - \frac{2\lambda_1}{\lambda_1^2-\lambda_1^{*2}} e^{\theta_1+\theta_1^*}$$

The 1-soliton solution can be expressed as below

$$u = \frac{-2ic_1(\lambda_1^2 - \lambda_1^{*2})\,exp\,(-\theta_1 + \theta_1^*)}{|c_1|^2 \lambda_1^*\,exp\,(-\theta_1 - \theta_1^*) + \lambda_1\,exp\,(\theta_1 + \theta_1^*)}. \quad (50)$$

Similarly, by expanding the determinant M and M_1 the 2-soliton solution can be expressed as

$$u = -4i\frac{det\,(M_1)_{3\times 3}}{det\,(M)_{2\times 2}} \quad (51)$$

Next, three coefficient functions δ(t) are selected to simulate the evolution of the soliton solutions.

Case 1: δ(t) = 1

The study set parameters $c_1 = 1, \lambda_1 = 1 + 0.5i$, and then depict the image of the 1-soliton solution in **Figure 1(a)**. Using parameters $c_1 = c_2 = 1, \lambda_1 = 1 + 0.5i, \lambda_2 = 0.8 + 0.4i$, the study displays the 2-soliton solution in **Figure 1(b)**.

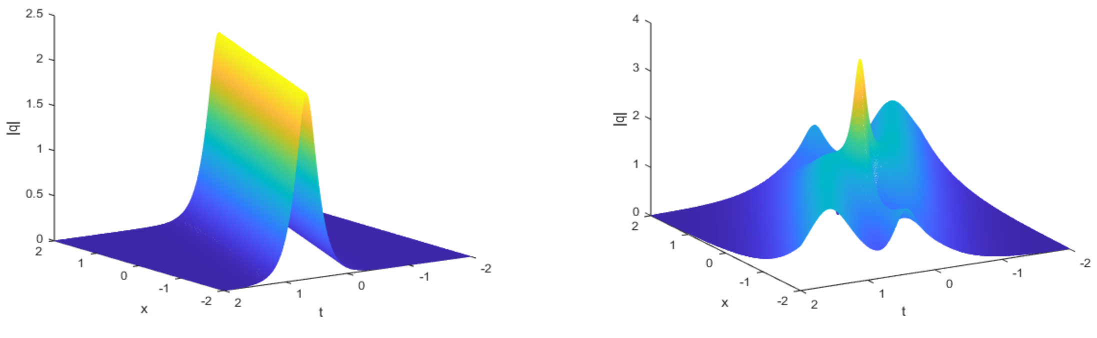
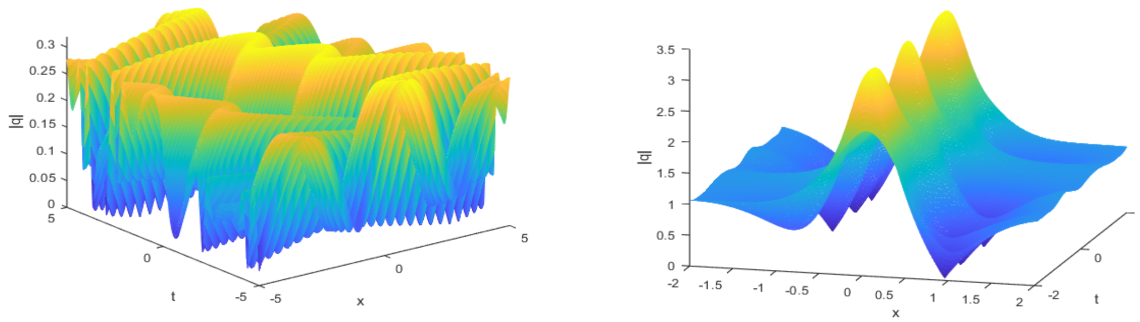


Figure 1. 3D plot of Equation 3: (a) 1-soliton solution; (b) 2-soliton solution

Case 2: $\delta(t) = cos^2(t)$

In **Figure 2(a)**, setting parameters $c_1 = c_2 = 1, \lambda_1 = 1 + 0.05i, \lambda_2 = 0.5 + 0.03i$, the study derive the structural characteristics of the mixed periodic solutions. **Figure 2(b)** shows the structural characteristics of the breather wave solutions with parameters $c_1 = c_2 = 1, \lambda_1 = 0.1 + 0.3i, \lambda_2 = 0.8 + 0.5i$.

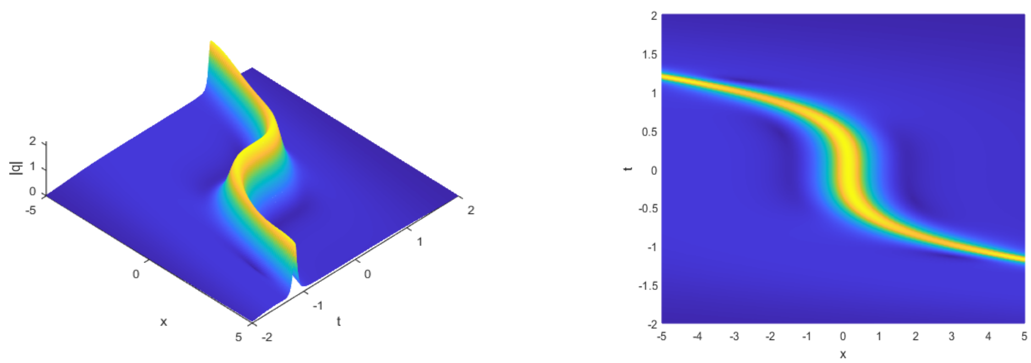


(a) Mixed periodic solution (b) Breather wave solution

Figure 2. Soliton solutions of Equation 3

Case 3: $\delta(t) = t^2$

Figure 3(a) displays a bright soliton solution with better structural characteristics when parameters are selected. **Figure 3(b)** shows the density plot of bright solitons.



(a) 3D plot (b) Density plot

Figure 3. Bright solitary wave solution of Equation 3

6. Conclusion

This paper uses the Riemann-Hilbert method to solve **Equation 3** and successfully provides a determinant representation of N-soliton solutions. Initially, the study formulated the RH problem for the variable coefficient nonlocal GI equation. Specifically, this study focuses on simple zero-point problems of scattering matrices. Considering that more zeros inevitably lead to more solutions, through spectral analysis, the study found that zeros exist in pairs. Then, the study obtained a solvable regularized RH problem by transformation. Moreover, the study selected three coefficient functions $\delta(t)$ to simulate the evolution of the soliton solutions.

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