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Projection Methods for Exponentially Regularized Nonconvex Variational Inequalities

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Abstract: This paper proposes a new exponentially regularized nonconvex variational inequality. The paper shows that the inequalities are equivalent to fixed point problems through the use of the projection properties. The paper develops a new self-adaptive projection iterative algorithm that not only dynamically adjusts the step size but also optimizes the projection direction, thereby improving the stability and convergence of the algorithm.

Keywords: Nonconvex variational inequality; Uniformly r-prox-regular fixed point problems; Self-adaptive finite p-step projection algorithm

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1. Introduction

Variational inequalities, first introduced by Stampacchia, have their theoretical origins in an in-depth study and generalization of classical variational problems [1]. It provides a direct, simple, and unified framework for addressing a wide range of complex problems. Since the mid-20th century, the applications of variational inequalities have become powerful mathematical tools for solving equilibrium problems. They have been referenced in fields such as economics, contact and friction problems, engineering, control and physics problems, transportation, optimization, and both pure and applied sciences [2–5].

In the early days of variational inequalities, most of the work on their existence and iterative methods was initially concentrated in the classical convex framework. However, continuity cannot be guaranteed, which makes it difficult to guarantee the existence and uniqueness of the solution to nonconvex variational inequalities. Bounkhel and Noor proposed some new variational inequalities, especially for the uniformly r-prox-regular set, which is a nonconvex set, but it can be regarded as a convex set in special circumstances [6–8]. Dupuis and Nagurney introduced and studied the projection dynamics system related to variational inequalities through an equivalent fixed point formula, providing a new approach to solving nonconvex problems [9]. It is worth mentioning that Bernstein was the first to study exponential convex functions [10]. Noor et al. further extended the classification of variational inequalities, integrating traditional variational inequalities and nonconvex variational inequalities [11–13].

Inspired by the previous works, this paper introduces and investigates a new exponentially regularized nonconvex variational inequality. Using projection techniques, we prove the equivalence between the exponentially regularized nonconvex variational inequality and fixed point problems. Based on this equivalence, the paper further explores the existence and uniqueness of solutions to the exponentially regularized nonconvex variational inequality. This study combines self-adaptive methods to propose a self-adaptive projection iteration and provide the iteration step size. The aim is to enhance the accuracy and stability of the projection direction, especially in complex applications. Finally, the convergence of the proposed iterative algorithm is thoroughly analyzed under suitable conditions. The results of this paper not only extend existing research but also offer significant improvements.

2. Preliminaries and basic results

Throughout this article, let H denote a real Hilbert space that is equipped with an inner product $<\bullet,\bullet>$ and corresponding norm $\|\bullet\|$. Also let K be a closed subset of H. Recall the following definitions and results of nonlinear convex analysis and non-smooth analysis [3, 10, 14–16].

Definition 1. Let $x \in H$ be a point that is not lying in K. The distance from x to $u \in K$ is called the closest distance or a projection of x onto K, it is expressed in the following formula

$$d_K(x) := \inf_{u \in K} \|x-u\| \cdot$$

The set of all the closest points is denoted by

$$P_K(x) := \{x \in H : \|x-u\| = d_K(x)\} .$$

Definition 2. The proximal normal cone of K is denoted by

$$N_K^P(u) := \{\xi \in H \mid u \in P_K(u+\alpha\xi), \alpha > 0\} \cdot$$

Lemma 1. Let K be a nonempty closed subset in H Then $\xi \in N_K^P(u)$ if and only if there exists a constant $\alpha = \alpha(\xi, u) > 0$ such that the following proximal normal inequality holds with all $v \in K$,

$$\langle \xi, v-u \rangle \le \alpha \|v-u\|^2 .$$

Definition 3. For any $r \in [0,+\infty]$, the subset $K_r \in H$ is called uniformly r-prox-regular. If every nonzero proximal normal to K_r can be implemented by an r-ball. This means that for all $x, \bar{x} \in K_r$ and $0 \ne \xi \in N_K^P(\bar{x})$

$$\langle \zeta, x-\bar{x} \rangle \le \frac{1}{2r} \|x-\bar{x}\|^2 \cdot$$

Lemma 2. A closed set $K \subset H$ is convex if and only if it is uniformly r-prox-regular set of radius r for every r>0.

Lemma 3. Let r>0 and K_r be a nonempty closed and uniformly r-prox-regular subset H. Set $U(r) = \{u \in H : 0 < d_{K_r} < r\}$. Then the following statements hold:

(1) For all $x \in U(r)$, $P_{K_r}(x) \neq \emptyset$;

(2) For all $r' \in (0,r)$, P_{K_r} is Lipschitz continuous with constant $L_P = \dfrac{r}{r-r'}$ on $U(r') = \{u \in H : 0 < d_{K_r} < r'\}$.

3. Projection methods and convergence analysis

In this section, with a given nonlinear operator $T,g:H\to H$ such that $K_r \subseteq g(H)$, the study will consider the problem of finding $u \in H$ such that $g(u) \in K_r$ and

$$\left\langle \rho e^{T(u)}, g(v)-g(u) \right\rangle + \frac{1}{2r}\left\|g(v)-g(u)\right\|^2 \geq 0, \quad \forall v \in H : g(v) \in K_r \cdot \tag{1}$$

The problem (1) is called the exponentially regularized nonconvex variational inequalities (ERNVID). The study gives the equivalence among ERNVID (1), problem (2), and the fixed point problem (3).

In the next proposition, the equivalence between nonconvex variational inclusion and exponentially regularized nonconvex variational inequalities (1) is established.

Proposition 1. If K_r is a uniformly r-prox-regular set, then problem (1) is equivalent to that of finding $g(u) \in K_r$ such that

$$0 \in \rho e^{T(u)} + N^P_{K_r}(g(u)) \cdot \tag{2}$$

The problem (2) is called the exponentially regularized nonconvex variational inclusion associated with the ERNVID problem.

Proof Let $g(u) \in K_r$ is the solution of the problem (1). If $\rho^{T(u)}=0$, it is known that vector zero always belongs to any normal cone, then

$$0 \in \rho e^{T(u)} + N^P_{K_r}(g(u)) \cdot$$

If $\rho^{T(u)} \neq 0$, one has

$$\left\langle -\rho e^{T(u)}, g(v)-g(u) \right\rangle \leq \frac{1}{2r}\left\|g(v)-g(u)\right\|^2 .$$

By Lemma 1,

$$0 \in \rho e^{T(u)} + N^P_{K_r}(g(u)) \cdot$$

Conversely, if $g(u) \in K_r$ is a solution of problem (2), then Definition 3 guarantees that $g(u) \in K_r$ is a solution to problem (1).

Lemma 4. Let T be the same as in the problem (1) and let ρ satisfy $0 < \rho < \dfrac{r'}{1+\left\|e^{T(u)}\right\|}$ for all $r' \in (0,r)$. Then $u \in H$ with $g(u) \in K_r$ is a solution of the problem (1) if and only if

$$g(u) = P_{K_r}(g(u) - \rho e^{T(u)}) , \tag{3}$$

where P_{K_r} is the projection of H onto K_r.

Proof Since $0 < \rho < \dfrac{r'}{1+\left\|e^{T(u)}\right\|}$, we have

$$0 \in \rho e^{T(u)} + N^P_{K_r}(g(u)) \Leftrightarrow g(u) - \rho e^{T(u)} = g(u) + N^P_{K_r}(g(u))$$
$$\Leftrightarrow g(u) - \rho e^{T(u)} = (I + N^P_{K_r})(g(u))$$
$$\Leftrightarrow g(u) = P_{K_r}(g(u) - \rho e^{T(u)}),$$

where I is the identity operator and the study has used the well-known fact that $P_{K_r} = (I + N^P_{K_r})^{-1}$.

4. New self-adaption algorithm

In this section, we propose a new adaptive iterative projection algorithm and provide a specific formula for the adaptive step size. These algorithms are schemes for finding the unique solution of the exponentially regularized nonconvex variational inequalities (ERNVID).

Algorithm

Let $u \in K_r$ be a fixed point and let $u_0 \in K_r$ be arbitrary. Iterative step: for $n \geq 0$, if

$$\left\| g(u_n) - P_{K_r}(g(u_n) - \rho e^{T(u_n)}) \right\| = 0,$$

then stop; otherwise the next $g(u_{n+1})$ by the following form

$$\begin{cases} g(u_{n+1}) = g(u_n) - \beta_n y_n \\ y_n = g(u_n) - P_{K_r}(g(u_n) - \rho e^{T(u_n)}) \end{cases}, \tag{4}$$

thereβ_n is chosen self-adaptively as

$$\beta_n = \max\left\{ \beta_{\min}, \frac{\left\| g(u_n) - P_{K_r}(g(u_n) - \rho e^{T(u_n)}) \right\|^2}{\left\| g(u_n) - P_{K_r}(g(u_n) - \rho e^{T(u_n)}) \right\|^2 + \left\| \rho e^{T(u_n)} + g(u_n) \right\|^2} \right\}.$$

Theorem 1. It is assumed that

(1) P_{K_r} is L_P-Lipschitzian with $L_P = \frac{r}{r - r'}$;

(2) e^T is L_e-Lipschitzian;

(3) g is G-strongly monotonic and Lipschitzian with $L_e \leq G$;

(4) β_n has a minimum value β_{min};

(5) ensure $\rho \in \frac{G}{L_e}(1 - \frac{\sqrt{2}-1}{L_P}, 1 + \frac{\sqrt{2}-1}{L_P})$.

Then the sequence u_n generated by (4) converges strongly to the solution u^* of the problem (1).

Proof From (4), the study has

$$\left\| g(u_{n+1}) - g(u^*) \right\|^2$$
$$= \left\| g(u_n) - g(u^*) \right\|^2 - 2\beta_n \left\langle g(u_n) - g(u^*), g(u_n) - P_{K_r}(g(u_n) - \rho e^{T(u_n)}) \right\rangle$$
$$+ \beta_n^2 \left\| g(u_n) - P_{K_r}(g(u_n) - \rho e^{T(u_n)}) \right\|^2,$$

let $\left\|g(u_{n+1})-g(u^*)\right\|=\left\|\ddot{0}_{n+1}\right\|$, then

$$\left\|\ddot{0}_{n+1}\right\|^2-\left\|\ _n\right\|^2=-2\beta_n\left\langle g(u_n)-g(u^*),g(u_n)-P_{K_r}(g(u_n)-\rho e^{T(u_n)})\right\rangle \qquad (5)$$

$$+\beta_n{}^2\left\|g(u_n)-P_{K_r}(g(u_n)-\rho e^{T(u_n)})\right\|^2.$$

Because the P_{K_r} is L_P-Lipschitzian,

$$\left\|P_{K_r}(g(u_n)-\rho e^{T(u_n)})-P_{K_r}(g(u^*)-\rho e^{T(u^*)})\right\|^2$$

$$\leq L_P{}^2\left\|\ddot{0}_n-\rho(e^{T(u_n)}-e^{T(u^*)})\right\|^2,$$

by use $g(u^*)=P_{K_r}(g(u^*)-\rho e^{T(u^*)})$, we have

$$\left\|P_{K_r}(g(u_n)-\rho e^{T(u_n)})-g(u^*)\right\|^2\leq L_P{}^2\left\|\ddot{0}_n-\rho(e^{T(u_n)}-e^{T(u^*)})\right\|^2,$$

then the following is obtained

$$\left\|P_{K_r}(g(u_n)-\rho e^{T(u_n)})-g(u^*)\right\|^2$$

$$\leq L_P{}^2\left\|\ddot{0}_n-\rho(e^{T(u_n)}-e^{T(u^*)})\right\|^2$$

$$\leq L_P{}^2(\left\|\ddot{0}_n\right\|^2+\rho^2\left\|e^{T(u_n)}-e^{T(u^*)}\right\|^2-2\rho\left\langle\ _n,e^{T(u_n)}-e^{T(u^*)}\right\rangle)$$

$$\leq L_P{}^2(\left\|\ddot{0}_n\right\|^2+\rho^2\left\|e^{T(u_n)}-e^{T(u^*)}\right\|^2-2\rho\left\|\ _n\right\|\left\|e^{T(u_n)}-e^{T(u^*)}\right\|).$$

The e^T is L_e-Lipschitzian, so

$$\left\|e^{T(u_n)}-e^{T(u^*)}\right\|\leq L_e\left\|u_n-u^*\right\|,$$

and g is G-strongly monotonic, so

$$\left\|g(u_n)-g(u^*)\right\|\geq G\left\|u_n-u^*\right\|,$$

because there has $L_e \leq G$, the following is obtained

$$\left\|e^{T(u_n)}-e^{T(u^*)}\right\|\leq\frac{L_e}{G}\left\|\ddot{0}_n\right\|.$$

So

$$\left\|P_{K_r}(g(u_n)-\rho e^{T(u_n)})-g(u^*)\right\|^2 \qquad (6)$$

$$\leq(1+\rho^2\frac{L_e{}^2}{G^2}-2\rho\frac{L_e}{G})L_P{}^2\left\|\ddot{0}_n\right\|^2$$

$$\leq C_1L_P{}^2\left\|\ddot{0}_n\right\|^2.$$

Then

$$\left\langle \ddot{o}_n, g(u_n) - P_{K_r}(g(u_n) - \rho e^{T(u_n)}) \right\rangle$$
$$= \left\langle \ddot{o}_n, \ _n - P_{K_r}(g(u_n) - \rho e^{T(u_n)}) - g(u^*) \right\rangle$$
$$= \|\ddot{o}_n\|^2 - \left\langle \ _n, P_{K_r}(g(u_n) - \rho e^{T(u_n)}) - g(u^*) \right\rangle$$
$$\geq \|\ddot{o}_n\|^2 - \|\ _n\| \left\| P_{K_r}(g(u_n) - \rho e^{T(u_n)}) - g(u^*) \right\|,$$

by use (6), the study has

$$\left\langle \ddot{o}_n, g(u_n) - P_{K_r}(g(u_n) - \rho e^{T(u_n)}) \right\rangle \geq (1 - \sqrt{C_1} L_P) \|\ _n\|^2 .$$

According to the definition

$$\beta_n = \frac{\left\| g(u_n) - P_{K_r}(g(u_n) - \rho e^{T(u_n)}) \right\|^2}{\left\| g(u_n) - P_{K_r}(g(u_n) - \rho e^{T(u_n)}) \right\|^2 + \left\| \rho e^{T(u_n)} + g(u_n) \right\|^2},$$

the study can get

$$\beta_n{}^2 \left\| g(u_n) - P_{K_r}(g(u_n) - \rho e^{T(u_n)}) \right\|^2$$
$$= \beta_n \frac{\left\| g(u_n) - P_{K_r}(g(u_n) - \rho e^{T(u_n)}) \right\|^4}{\left\| g(u_n) - P_{K_r}(g(u_n) - \rho e^{T(u_n)}) \right\|^2 + \left\| \rho e^{T(u_n)} + g(u_n) \right\|^2}$$
$$\leq \beta_n \left\| g(u_n) - P_{K_r}(g(u_n) - \rho e^{T(u_n)}) \right\|^2$$
$$\leq \beta_n (1 + C_1 L_P^2) \|\ddot{o}_n\|^2 .$$

Substituting into (5), the study obtains

$$\|\ddot{o}_{n+1}\|^2 - \|\ _n\|^2 \leq -\beta_n C_2 \|\ _n\|^2 ,$$

here $1 - 2\sqrt{C_1} L_P - C_1 L_P^2 = C_2$.

Summing up both sides

$$\sum_{n=0}^{N} (\|\ddot{o}_{n+1}\|^2 - \|\ _n\|^2) \leq -\sum_{n=0}^{N} \beta_n C_2 \|\ _n\|^2 ,$$

$$\|\ddot{o}_{n+1}\|^2 - \|\ _0\|^2 \leq -\sum_{n=0}^{N} \beta_n C_2 \|\ _n\|^2 ,$$

with $\|\ddot{o}_{n+1}\|^2 \geq 0$,

$$\sum_{n=0}^{N} \beta_n C_2 \|\ddot{o}_n\|^2 \leq \|\ _0\|^2 ,$$

to ensure $\left\|\ddot{0}_{n+1}\right\|^2 \to 0$, the study assumes that there exists β_{min},

$$\beta_n = \max\left\{\beta_{\min}, \frac{\left\|g(u_n) - P_{K_r}(g(u_n) - \rho e^{T(u_n)})\right\|^2}{\left\|g(u_n) - P_{K_r}(g(u_n) - \rho e^{T(u_n)})\right\|^2 + \left\|\rho e^{T(u_n)} + g(u_n)\right\|^2}\right\},$$

so

$$\left\|g(u_n) - g(u^*)\right\| \to 0 \quad (n \to \infty).$$

Because g is strongly monotonic,

$$\left\|u_n - u^*\right\| \to 0 \quad (n \to \infty),$$

which ends the proof.

The new adaptive iteration and adaptive step size were given to control the gradient direction, ensure the update of the convergence direction of the algorithm.

5. Conclusion

In this paper, the authors have introduced and investigated a new exponentially regularized nonconvex variational inequality. By employing projection operator techniques, the study establishes the equivalence between exponentially regularized nonconvex variational inequalities and fixed point problems. This study combines the adaptive method with the iterative framework and proposes a new adaptive projection iterative algorithm based on the given projection method, aiming to improve the efficiency and stability of solving variational inequalities, especially in complex problem settings. Finally, the study provides a detailed convergence analysis of the proposed iterative algorithm under suitable conditions. The findings of this study not only expand the existing theoretical results but also make significant improvements. Interested readers can further explore the broad applications and other properties of exponentially regularized nonconvex variational inequalities in pure and applied sciences, presenting an exciting direction for future research.

Disclosure statement

The authors declare no conflict of interest.

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