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De Moivre's Theorem for the Matrix Representation of Dual Generalized Quaternions

Xiangqiang Kong*

Department of Mathematics and Statistics, Heze University, Heze 274015, Shandong Province, China

****Corresponding author:*** Xiangqiang Kong, kong3058@126.com

Abstract: In this paper, based on the concept of dual generalized quaternions, the study of dual generalized quaternions is transformed into a study of the matrix representation of dual generalized quaternions. With the aid of a polar representation for dual generalized quaternions, De Moivre's theorem is obtained for the matrix representation of dual generalized quaternions, and Euler's formula is extended. The relations between the powers of matrices associated with dual generalized quaternions are determined, and the n-th root of the matrix representation equation of dual generalized quaternions is found.

Keywords: Dual generalized quaternion; Matrix representation; De Moivre's theorem; Euler's formula

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1. Introduction

In 1843, Hamilton discovered quaternions, in which they are widely used in differential geometry, computer graphics, quantum physics, and space kinematics. Cockle introduced the set of split quaternions in 1849. Unlike quaternions, the set of split quaternions is four-dimensional noncommutative, which has zero divisors, nilpotent elements, and idempotent elements [1-3]. Clifford introduced dual numbers, which are extensions of real numbers by adjoining a new element ε, where $\varepsilon^2=0$ [4]. In several studies [5-7], dual quaternions and their applications were considered. In other studies [8-14], generalized quaternions and their properties were studied, in which De Moivre's theorem and Euler's formula were obtained for generalized quaternions. In 2020, Kong studied commutative quaternions and split semi quaternions, and De Moivre's theorem was obtained for the matrix representation of split (semi) quaternions [15-17]. The research on dual generalized quaternions has achieved certain results, with De Moivre's theorem obtained for dual generalized quaternions [18,19].

In this paper, the focus is on De Moivre's theorem for the matrix representation of generalized dual quaternions. The basic concepts of dual numbers, generalized quaternions, and dual generalized quaternions are reviewed in the second section. The third section shows the matrix representation of dual generalized quaternions, and some properties of the matrix representation are obtained. In the fourth section, with the matrix representation of dual generalized quaternions, the De Moivre's theorem and Euler's formula for the matrix representation of dual generalized quaternions in different cases are obtained.

2. Preliminaries

(1) **Definition 1**

A dual number A has the form $t+\varepsilon t^*$, where t and t^* are real numbers, ε is the dual symbol subjected to the rulers:

$$\varepsilon \neq 0;\ \ \varepsilon^2 = 0;\ \ 0\cdot\varepsilon = \varepsilon\cdot 0 = 0;\ \ 1\cdot\varepsilon = \varepsilon\cdot 1 = \varepsilon.$$

The set of dual numbers is denoted by **D**. The summation and multiplication of two dual numbers are defined as similar to complex numbers, but it must be forgotten that $\varepsilon^2 = 0$. So, **D** is a commutative ring, which has a unit element. The parameters t and t^* are the real and dual parts of A, respectively. The addition and multiplication on two dual real numbers $A = t + \varepsilon t^*$ and $B = c + \varepsilon c^*$ are as follows:

$$A + B = \varepsilon\left(t^* + c^*\right) + \left(t + c\right);\ \ AB = \varepsilon\left(tc^* + t^*c\right) + tc.$$

Dual number $A = t + \varepsilon t^*\,(t \neq 0)$ has a multiplicative inverse, which is $A^{-1} = \frac{1}{t} - \frac{t^*}{t^2}\varepsilon$. If $t = 0$, εt^* is called pure dual number. The conjugate of the dual number $A = t + \varepsilon t^*$ is $\bar{A} = t - \varepsilon t^*$.

(2) **Definition 2**

A generalized quaternion *s* is an expression of

$$s = s_0 + s_1 i + s_2 j + s_3 k,$$

s_0, s_1, s_2 and s_3 are real numbers; *i*, *j*, and *k* are quaternionic units, which satisfy the equalities

$$i^2 = -\alpha,\ \ j^2 = -\beta,\ \ k^2 = -\alpha\beta$$

$$ij = k = -ji,\ \ jk = \beta i = -kj$$

$$ki = \alpha j = -ik,\ \ \alpha, \beta \in \mathbf{R}.$$

The set of generalized quaternions is denoted by $\mathbf{H}_{\alpha\beta}$. If $\alpha = \beta = 1$, $\mathbf{H}_{\alpha\beta}$ is real quaternions. If $\alpha = 1, \beta = -1$, $\mathbf{H}_{\alpha\beta}$ is split quaternions. The addition and multiplication for two generalized quaternions $s = s_0 + s_1 i + s_2 j + s_3 k$ and $t = t_0 + t_1 i + t_2 j + t_3 k$ are defined as

$$s + t = \left(s_0 + t_0\right) + \left(s_1 + t_1\right)i + \left(s_2 + t_2\right)j + \left(s_3 + t_3\right)k,$$

$$st = \left(s_0t_0 - \alpha s_1t_1 - \beta s_2t_2 - \alpha\beta s_3t_3\right) + \left(s_0t_1 + s_1t_0 - \beta s_2t_3 + \beta s_3t_2\right)i$$
$$+ \left(s_0t_2 + \alpha s_1t_3 + s_2t_0 - \alpha s_3t_1\right)j + \left(s_0t_3 - s_1t_2 + s_2t_1 + s_3t_0\right)k.$$

(3) **Definition 3**

A dual generalized quaternion $\hat{q}$ is written as

$$\hat{q} = \hat{q}_0 + \hat{q}_1 i + \hat{q}_2 j + \hat{q}_3 k,$$

$\hat{q}_0, \hat{q}_1, \hat{q}_2$ and $\hat{q}_3$ are dual numbers; *i*, *j*, and *k* are quaternionic units, which satisfy in the above equalities. The set of dual generalized quaternions is denoted by $\tilde{\mathbf{H}}_{\alpha\beta}$. The dual generalized quaternion $\hat{q}$ can also be written as follows:

$$\hat{q} = q + \varepsilon q^{*},$$

where q is the real part and q^{*} is the pure generalized dual quaternion component. The dual generalized quaternion $\hat{q} = \hat{q}_0 + \hat{q}_1 i + \hat{q}_2 j + \hat{q}_3 k$ is divided into two parts: the scalar part $S_{\hat{q}} = \hat{q}_0$, and the vector part $\vec{V}_{\hat{q}} = \hat{q}_1 i + \hat{q}_2 j + \hat{q}_3 k$. Hence, it can be written, $\hat{q} = S_{\hat{q}} + \vec{V}_{\hat{q}}$. $\hat{q}$ is called pure generalized dual quaternion if $S_{\hat{q}} = 0$. The conjugate of $\hat{q} = S_{\hat{q}} + \vec{V}_{\hat{q}}$ is defined as $\bar{\hat{q}} = S_{\hat{q}} - \vec{V}_{\hat{q}}$. The norm of $\hat{q} = S_{\hat{q}} + \vec{V}_{\hat{q}}$ is defined as

$$N_{\hat{q}} = \hat{q}\bar{\hat{q}} = \bar{\hat{q}}\hat{q} = \hat{q}_0^2 + \alpha \hat{q}_1^2 + \beta \hat{q}_2^2 + \alpha\beta \hat{q}_3^2.$$

The norm of $\vec{V}_{\hat{q}}$:

$$N_{\vec{V}_{\hat{q}}} = \alpha \hat{q}_1^2 + \beta \hat{q}_2^2 + \alpha\beta \hat{q}_3^2.$$

If $N_{\hat{q}} = 1$, $\hat{q}$ is referred to as unit dual generalized quaternion. The addition and multiplication for two dual generalized quaternions $\hat{p} = \hat{p}_0 + \hat{p}_1 i + \hat{p}_2 j + \hat{p}_3 k$ and $\hat{q} = \hat{q}_0 + \hat{q}_1 i + \hat{q}_2 j + \hat{q}_3 k$ are defined as

$$\hat{p} + \hat{q} = (\hat{p}_0 + \hat{q}_0) + (\hat{p}_1 + \hat{q}_1) i + (\hat{p}_2 + \hat{q}_2) j + (\hat{p}_3 + \hat{q}_3) k.$$

$$\hat{p}\hat{q} = (\hat{p}_0\hat{q}_0 - \alpha \hat{p}_1\hat{q}_1 - \beta \hat{p}_2\hat{q}_2 - \alpha\beta \hat{p}_3\hat{q}_3) + (\hat{p}_0\hat{q}_1 + \hat{p}_1\hat{q}_0 - \beta \hat{p}_2\hat{q}_3 + \beta \hat{p}_3\hat{q}_2) i$$

$$+ (\hat{p}_0\hat{q}_2 + \alpha \hat{p}_1\hat{q}_3 + \hat{p}_2\hat{q}_0 - \alpha \hat{p}_3\hat{q}_1) j + (\hat{p}_0\hat{q}_3 - \hat{p}_1\hat{q}_2 + \hat{p}_2\hat{q}_1 + \hat{p}_3\hat{q}_0) k.$$

Also, this can be written as follows:

$$\hat{q}\hat{p} = \begin{pmatrix} \hat{q}_0 & -\alpha\hat{q}_1 & -\beta\hat{q}_2 & -\alpha\beta\hat{q}_3 \\ \hat{q}_1 & \hat{q}_0 & -\beta\hat{q}_3 & \beta\hat{q}_2 \\ \hat{q}_2 & \alpha\hat{q}_3 & \hat{q}_0 & -\alpha\hat{q}_1 \\ \hat{q}_3 & -\hat{q}_2 & \hat{q}_1 & \hat{q}_0 \end{pmatrix} \begin{pmatrix} \hat{p}_0 \\ \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \end{pmatrix}.$$

3. The matrix representation of dual generalized quaternion

Theorem 3.1

Every dual generalized quaternion can be represented by a 4×4 dual matrix.

Proof

Assume $\hat{q} = \hat{q}_0 + \hat{q}_1 i + \hat{q}_2 j + \hat{q}_3 k$ as an element of $\hat{\mathbb{H}}_{\alpha\beta}$. The mapping $\tau : \hat{\mathbb{H}}_{\alpha\beta} \to \hat{\mathbb{H}}_{\alpha\beta}$ is written as $\tau_{\hat{q}}(\hat{p}) = \hat{q}\hat{p}$, $\forall \hat{p} \in \hat{\mathbb{H}}_{\alpha\beta}$. The mapping is defined as follows:

$$\tau_{\hat{q}}(1) = \hat{q}_0 + \hat{q}_1 i + \hat{q}_2 j + \hat{q}_3 k; \qquad \tau_{\hat{q}}(i) = -\alpha\hat{q}_1 + \hat{q}_0 i + \alpha\hat{q}_3 j - \hat{q}_2 k;$$
$$\tau_{\hat{q}}(j) = -\beta\hat{q}_2 - \beta\hat{q}_3 i + \hat{q}_0 j + \hat{q}_1 k; \qquad \tau_{\hat{q}}(k) = -\alpha\beta\hat{q}_3 + \beta\hat{q}_2 i - \alpha\hat{q}_1 j + \hat{q}_0 k.$$

According to the mapping, the set of dual generalized quaternions is defined as a subset of dual matrices

$$M_{4\times 4}(\mathbf{D})=\left\{\begin{pmatrix}\hat{q}_0 & -\alpha\hat{q}_1 & -\beta\hat{q}_2 & -\alpha\beta\hat{q}_3\\ \hat{q}_1 & \hat{q}_0 & -\beta\hat{q}_3 & \beta\hat{q}_2\\ \hat{q}_2 & \alpha\hat{q}_3 & \hat{q}_0 & -\alpha\hat{q}_1\\ \hat{q}_3 & -\hat{q}_2 & \hat{q}_1 & \hat{q}_0\end{pmatrix};\hat{q}_0,\hat{q}_1,\hat{q}_2,\hat{q}_3\in\mathbf{D}\right\}.$$

So, choosing $\tau_{\hat{q}}$ as the linear transformation on $\hat{\mathbf{H}}_{\alpha\beta}$ and $f:\hat{q}\rightarrow\tau_{\hat{q}}$ as the ring homomorphism between the rings of all linear transformations from $\hat{\mathbf{H}}_{\alpha\beta}$ to $\hat{\mathbf{H}}_{\alpha\beta}$. After selecting the basis $1,i,j,k$ of $\hat{\mathbf{H}}_{\alpha\beta}$, each linear transformation on $\hat{\mathbf{H}}_{\alpha\beta}$ can be equivalent to a 4×4 dual matrix, i.e., $f(\hat{\mathbf{H}}_{\alpha\beta})=M_{4\times 4}(\mathbf{D})$. Therefore, $f(\hat{\mathbf{H}}_{\alpha\beta})$ and $M_{4\times 4}(\mathbf{D})$ are isomorphic; $f(\hat{\mathbf{H}}_{\alpha\beta})$ and $M_{4\times 4}(\mathbf{D})$ are the same thing. So, the study of a dual generalized quaternion is equivalent to dealing with the corresponding dual matrix. Writing the dual matrix of any dual generalized quaternion $\hat{q}$ by $A_{\hat{q}}$, i.e.,

$$A_{\hat{q}}=\begin{pmatrix}\hat{q}_0 & -\alpha\hat{q}_1 & -\beta\hat{q}_2 & -\alpha\beta\hat{q}_3\\ \hat{q}_1 & \hat{q}_0 & -\beta\hat{q}_3 & \beta\hat{q}_2\\ \hat{q}_2 & \alpha\hat{q}_3 & \hat{q}_0 & -\alpha\hat{q}_1\\ \hat{q}_3 & -\hat{q}_2 & \hat{q}_1 & \hat{q}_0\end{pmatrix}.$$

The following properties can be verified by the above definition of $A_{\hat{q}}$.

Proposition 3.1
Let $\hat{q},\hat{p}\in\hat{\mathbf{H}}_{\alpha\beta}$, $\lambda\in\mathbf{R}$, the following can be obtained:

$$(i)A_{\hat{q}+\hat{p}}=A_{\hat{q}}+A_{\hat{p}},\qquad (ii)A_{\lambda\hat{q}}=\lambda A_{\hat{q}},\qquad (iii)A_{\hat{q}\hat{p}}=A_{\hat{q}}A_{\hat{p}},$$

$$(iv)\det\left(A_{\hat{q}}\right)=\left(N_{\hat{q}}\right)^2,\qquad (v)\operatorname{tr}\left(A_{\hat{q}}\right)=4\hat{q}_0,\qquad (vi)\hat{q}=\hat{p}\Leftrightarrow A_{\hat{q}}=A_{\hat{p}}.$$

4. De Moivre's theorem for the matrix representation of dual generalized quaternions

De Moivre's theorem for the representation matrix of dual generalized quaternions in different cases is investigated.

(1) **Case 1**

Assuming α,β are positive numbers.

Assuming $\hat{q}=\hat{q}_0+\hat{q}_1 i+\hat{q}_2 j+\hat{q}_3 k$ is a non-zero dual generalized quaternion, the polar form of $\hat{q}$ can be written $\hat{q}=\cos\hat{\theta}+\vec{s}\sin\hat{\theta}$ [18], where $\cos\hat{\theta}=\hat{q}_0$, $\sin\hat{\theta}=\sqrt{\alpha\hat{q}_1^2+\beta\hat{q}_2^2+\alpha\beta\hat{q}_3^2}$, $\hat{\theta}=\theta+\varepsilon\theta^*$, and the unit dual vector is $\vec{s}=\dfrac{\hat{q}_1 i+\hat{q}_2 j+\hat{q}_3 k}{\sqrt{\alpha\hat{q}_1^2+\beta\hat{q}_2^2+\alpha\beta\hat{q}_3^2}}=s_1 i+s_2 j+s_3 k=(s_1,s_2,s_3)$, $\alpha\hat{q}_1^2+\beta\hat{q}_2^2+\alpha\beta\hat{q}_3^2\neq 0$, $\vec{s}^2=-1$.

For the unit pure dual generalized quaternion $\vec{s} = s_1 i + s_2 j + s_3 k = (s_1, s_2, s_3)$, $\vec{s}^{\,2} = -1$, the matrix representation of $\vec{s}$ is $A_{\vec{s}} = \begin{pmatrix} 0 & -\alpha s_1 & -\beta s_2 & -\alpha\beta s_3 \\ s_1 & 0 & -\beta s_3 & \beta s_2 \\ s_2 & \alpha s_3 & 0 & -\alpha s_1 \\ s_3 & -s_2 & s_1 & 0 \end{pmatrix}$. Obtaining $A_{\vec{s}}^2 = -\boldsymbol{I}_4$, the generalization of Euler's formula for matrix $A_{\vec{s}}$ is as follows:

$$e^{A_{\vec{s}}\hat{\theta}} = \boldsymbol{I}_4 + A_{\vec{s}}\hat{\theta} + \frac{\left(A_{\vec{s}}\hat{\theta}\right)^2}{2!} + \frac{\left(A_{\vec{s}}\hat{\theta}\right)^3}{3!} + \frac{\left(A_{\vec{s}}\hat{\theta}\right)^4}{4!} + \frac{\left(A_{\vec{s}}\hat{\theta}\right)^5}{5!} + \cdots$$

$$= \boldsymbol{I}_4\left(1 - \frac{\hat{\theta}^2}{2!} + \frac{\hat{\theta}^4}{4!} - \cdots\right) + A_{\vec{s}}\left(\hat{\theta} - \frac{\hat{\theta}^3}{3!} + \frac{\hat{\theta}^5}{5!} - \cdots\right)$$

$$= \boldsymbol{I}_4 \cos\hat{\theta} + A_{\vec{s}} \sin\hat{\theta}$$

$$= \begin{pmatrix} \cos\hat{\theta} & -\alpha s_1 \sin\hat{\theta} & -\beta s_2 \sin\hat{\theta} & -\alpha\beta s_3 \sin\hat{\theta} \\ s_1 \sin\hat{\theta} & \cos\hat{\theta} & -\beta s_3 \sin\hat{\theta} & \beta s_2 \sin\hat{\theta} \\ s_2 \sin\hat{\theta} & \alpha s_3 \sin\hat{\theta} & \cos\hat{\theta} & -\alpha s_1 \sin\hat{\theta} \\ s_3 \sin\hat{\theta} & -s_2 \sin\hat{\theta} & s_1 \sin\hat{\theta} & \cos\hat{\theta} \end{pmatrix}.$$

Lemma 4.1

Let $\vec{s}$ be a unit pure dual generalized quaternion; if it satisfies $\vec{s} = -1$, then we have

$$\left(\cos\hat{a} + \vec{s}\sin\hat{a}\right)\left(\cos\hat{b} + \vec{s}\sin\hat{b}\right) = \cos\left(\hat{a} + \hat{b}\right) + \vec{s}\sin\left(\hat{a} + \hat{b}\right).$$

Proof

A similar proof can be seen in a study [6].

Theorem 4.1

Assuming $\hat{q} = \hat{q}_0 + \hat{q}_1 i + \hat{q}_2 j + \hat{q}_3 k$ is a unit dual generalized quaternion, the polar form $\hat{q} = e^{\vec{s}\hat{\theta}} = \cos\hat{\theta} + \vec{s}\sin\hat{\theta}$, where $N_{\vec{s}} = 1$. Then, for every integer n,

$$A_{\hat{q}} = \begin{pmatrix} \cos\hat{\theta} & -\alpha s_1 \sin\hat{\theta} & -\beta s_2 \sin\hat{\theta} & -\alpha\beta s_3 \sin\hat{\theta} \\ s_1 \sin\hat{\theta} & \cos\hat{\theta} & -\beta s_3 \sin\hat{\theta} & \beta s_2 \sin\hat{\theta} \\ s_2 \sin\hat{\theta} & \alpha s_3 \sin\hat{\theta} & \cos\hat{\theta} & -\alpha s_1 \sin\hat{\theta} \\ s_3 \sin\hat{\theta} & -s_2 \sin\hat{\theta} & s_1 \sin\hat{\theta} & \cos\hat{\theta} \end{pmatrix}, \tag{4.1}$$

The n-th power of the matrix $A_{\hat{q}}$ reads as

$$A_{\hat{q}}^{n}=\begin{pmatrix} \cos\left(n\hat{\theta}\right) & -\alpha s_1\sin\left(n\hat{\theta}\right) & -\beta s_2\sin\left(n\hat{\theta}\right) & -\alpha\beta s_3\sin\left(n\hat{\theta}\right) \\ s_1\sin\left(n\hat{\theta}\right) & \cos\left(n\hat{\theta}\right) & -\beta s_3\sin\left(n\hat{\theta}\right) & \beta s_2\sin\left(n\hat{\theta}\right) \\ s_2\sin\left(n\hat{\theta}\right) & \alpha s_3\sin\left(n\hat{\theta}\right) & \cos\left(n\hat{\theta}\right) & -\alpha s_1\sin\left(n\hat{\theta}\right) \\ s_3\sin\left(n\hat{\theta}\right) & -s_2\sin\left(n\hat{\theta}\right) & s_1\sin\left(n\hat{\theta}\right) & \cos\left(n\hat{\theta}\right) \end{pmatrix}.$$

Proof

Using induction on positive integer n. Assume that

$$A_{\hat{q}}^{n}=\begin{pmatrix} \cos\left(n\hat{\theta}\right) & -\alpha s_1\sin\left(n\hat{\theta}\right) & -\beta s_2\sin\left(n\hat{\theta}\right) & -\alpha\beta s_3\sin\left(n\hat{\theta}\right) \\ s_1\sin\left(n\hat{\theta}\right) & \cos\left(n\hat{\theta}\right) & -\beta s_3\sin\left(n\hat{\theta}\right) & \beta s_2\sin\left(n\hat{\theta}\right) \\ s_2\sin\left(n\hat{\theta}\right) & \alpha s_3\sin\left(n\hat{\theta}\right) & \cos\left(n\hat{\theta}\right) & -\alpha s_1\sin\left(n\hat{\theta}\right) \\ s_3\sin\left(n\hat{\theta}\right) & -s_2\sin\left(n\hat{\theta}\right) & s_1\sin\left(n\hat{\theta}\right) & \cos\left(n\hat{\theta}\right) \end{pmatrix},$$

The following formula is true based on **Lemma 4.1**.

$$A_{\hat{q}}^{n+1}=\begin{pmatrix} \cos\left(\left(n+1\right)\hat{\theta}\right) & -\alpha s_1\sin\left(\left(n+1\right)\hat{\theta}\right) & -\beta s_2\sin\left(\left(n+1\right)\hat{\theta}\right) & -\alpha\beta s_3\sin\left(\left(n+1\right)\hat{\theta}\right) \\ s_1\sin\left(\left(n+1\right)\hat{\theta}\right) & \cos\left(\left(n+1\right)\hat{\theta}\right) & -\beta s_3\sin\left(\left(n+1\right)\hat{\theta}\right) & \beta s_2\sin\left(\left(n+1\right)\hat{\theta}\right) \\ s_2\sin\left(\left(n+1\right)\hat{\theta}\right) & \alpha s_3\sin\left(\left(n+1\right)\hat{\theta}\right) & \cos\left(\left(n+1\right)\hat{\theta}\right) & -\alpha s_1\sin\left(\left(n+1\right)\hat{\theta}\right) \\ s_3\sin\left(\left(n+1\right)\hat{\theta}\right) & -s_2\sin\left(\left(n+1\right)\hat{\theta}\right) & s_1\sin\left(\left(n+1\right)\hat{\theta}\right) & \cos\left(\left(n+1\right)\hat{\theta}\right) \end{pmatrix}.$$

The theorem holds for all integer n, since

$$A_{\hat{q}}^{-n}=\begin{pmatrix} \cos\left(n\hat{\theta}\right) & \alpha s_1\sin\left(n\hat{\theta}\right) & \beta s_2\sin\left(n\hat{\theta}\right) & \alpha\beta s_3\sin\left(n\hat{\theta}\right) \\ -s_1\sin\left(n\hat{\theta}\right) & \cos\left(n\hat{\theta}\right) & \beta s_3\sin\left(n\hat{\theta}\right) & -\beta s_2\sin\left(n\hat{\theta}\right) \\ -s_2\sin\left(n\hat{\theta}\right) & -\alpha s_3\sin\left(n\hat{\theta}\right) & \cos\left(n\hat{\theta}\right) & \alpha s_1\sin\left(n\hat{\theta}\right) \\ -s_3\sin\left(n\hat{\theta}\right) & s_2\sin\left(n\hat{\theta}\right) & -s_1\sin\left(n\hat{\theta}\right) & \cos\left(n\hat{\theta}\right) \end{pmatrix}.$$

Let $\hat{q}=\hat{q}_0+\hat{q}_1 i+\hat{q}_2 j+\hat{q}_3 k$, a unit dual generalized quaternion, and the polar form is $\hat{q}=\cos\hat{\theta}+\vec{s}\sin\hat{\theta}$, where $\hat{\theta}=\theta\in\mathbf{R}$ $(\theta^*=0)$. The matrix associated with $\hat{q}$ is of the form (4.1). In general case, the matrix (4.1) is obtained by

$$A_{\hat{q}}=\begin{pmatrix}\cos(\theta+2t\pi) & -\alpha s_1\sin(\theta+2t\pi) & -\beta s_2\sin(\theta+2t\pi) & -\alpha\beta s_3\sin(\theta+2t\pi)\\ s_1\sin(\theta+2t\pi) & \cos(\theta+2t\pi) & -\beta s_3\sin(\theta+2t\pi) & \beta s_2\sin(\theta+2t\pi)\\ s_2\sin(\theta+2t\pi) & \alpha s_3\sin(\theta+2t\pi) & \cos(\theta+2t\pi) & -\alpha s_1\sin(\theta+2t\pi)\\ s_3\sin(\theta+2t\pi) & -s_2\sin(\theta+2t\pi) & s_1\sin(\theta+2t\pi) & \cos(\theta+2t\pi)\end{pmatrix},$$

where $k\in\mathbf{Z}$. The equation $X^n=A_{\hat{q}}$ has n roots, and they are as follows:

$$A_k^{\frac{1}{n}}=\begin{pmatrix}\cos\left(\frac{\theta+2k\pi}{n}\right) & -\alpha s_1\sin\left(\frac{\theta+2k\pi}{n}\right) & -\beta s_2\sin\left(\frac{\theta+2k\pi}{n}\right) & -\alpha\beta s_3\sin\left(\frac{\theta+2k\pi}{n}\right)\\ s_1\sin\left(\frac{\theta+2k\pi}{n}\right) & \cos\left(\frac{\theta+2k\pi}{n}\right) & -\beta s_3\sin\left(\frac{\theta+2k\pi}{n}\right) & \beta s_2\sin\left(\frac{\theta+2k\pi}{n}\right)\\ s_2\sin\left(\frac{\theta+2k\pi}{n}\right) & \alpha s_3\sin\left(\frac{\theta+2k\pi}{n}\right) & \cos\left(\frac{\theta+2k\pi}{n}\right) & -\alpha s_1\sin\left(\frac{\theta+2k\pi}{n}\right)\\ s_3\sin\left(\frac{\theta+2k\pi}{n}\right) & -s_2\sin\left(\frac{\theta+2k\pi}{n}\right) & s_1\sin\left(\frac{\theta+2k\pi}{n}\right) & \cos\left(\frac{\theta+2k\pi}{n}\right)\end{pmatrix},$$

where $k=0,1,2,\ldots,n-1$.

Example 4.1

Let $\hat{q}=\frac{1}{2}+\frac{1}{2\sqrt{\alpha}}i+\frac{1}{\sqrt{2\beta}}j+\varepsilon k$, as a unit dual generalized quaternion, the polar form $\hat{q}=\cos\frac{\pi}{3}+\vec{s}\sin\frac{\pi}{3}$, where $\vec{s}=\frac{1}{\sqrt{3\alpha}}i+\frac{2}{\sqrt{6\beta}}j+\frac{2\varepsilon}{\sqrt{3}}k=(s_1,s_2,s_3)$, then

$$A_{\hat{q}}=\begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{\alpha}}{2} & -\sqrt{\frac{\beta}{2}} & -\alpha\beta\varepsilon\\ \frac{1}{2\sqrt{\alpha}} & \frac{1}{2} & -\beta\varepsilon & \sqrt{\frac{\beta}{2}}\\ \frac{1}{\sqrt{2\beta}} & \alpha\varepsilon & \frac{1}{2} & -\frac{\sqrt{\alpha}}{2}\\ \varepsilon & -\frac{1}{\sqrt{2\beta}} & \frac{1}{2\sqrt{\alpha}} & \frac{1}{2}\end{pmatrix},$$

The equation $X^2 = A_{\hat{q}}$ has two roots, and the square roots of $A_{\hat{q}}$ can be obtained.
The first root for $k=0$ is

$$A_0^{\frac{1}{2}} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{\sqrt{\alpha}}{2\sqrt{3}} & -\sqrt{\frac{\beta}{6}} & -\frac{\alpha\beta\varepsilon}{\sqrt{3}} \\ \frac{1}{2\sqrt{3\alpha}} & \frac{\sqrt{3}}{2} & -\frac{\beta\varepsilon}{\sqrt{3}} & \sqrt{\frac{\beta}{6}} \\ \frac{1}{\sqrt{6\beta}} & \frac{\alpha\varepsilon}{\sqrt{3}} & \frac{\sqrt{3}}{2} & -\frac{\sqrt{\alpha}}{2\sqrt{3}} \\ \frac{\varepsilon}{\sqrt{3}} & -\frac{1}{\sqrt{6\beta}} & \frac{1}{2\sqrt{3\alpha}} & \frac{\sqrt{3}}{2} \end{pmatrix},$$

and the second one for $k=1$ is

$$A_1^{\frac{1}{2}} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{\sqrt{\alpha}}{2\sqrt{3}} & \sqrt{\frac{\beta}{6}} & \frac{\alpha\beta\varepsilon}{\sqrt{3}} \\ -\frac{1}{2\sqrt{3\alpha}} & -\frac{\sqrt{3}}{2} & \frac{\beta\varepsilon}{\sqrt{3}} & -\sqrt{\frac{\beta}{6}} \\ -\frac{1}{\sqrt{6\beta}} & -\frac{\alpha\varepsilon}{\sqrt{3}} & -\frac{\sqrt{3}}{2} & \frac{\sqrt{\alpha}}{2\sqrt{3}} \\ -\frac{\varepsilon}{\sqrt{3}} & \frac{1}{\sqrt{6\beta}} & -\frac{1}{2\sqrt{3\alpha}} & -\frac{\sqrt{3}}{2} \end{pmatrix}.$$

Also, it is easy to see that $A_0^{\frac{1}{2}} + A_1^{\frac{1}{2}} = 0$.

Theorem 4.2
Assuming $\hat{q}$ is a unit dual generalized quaternion, the polar form $\hat{q} = \cos\theta + \overset{1}{s}\sin\theta$, where $\theta \in \mathbf{R}$. Assuming that $t = \frac{2\pi}{\theta} \in \mathbf{Z}^+ - \{1\}$, we can obtain $n = p(\text{mod} \quad t)$ if and only if $A_{\hat{q}}^n = A_{\hat{q}}^p$.

Proof
Let $n = p(\text{mod} \quad t)$. So, $n = c \cdot t + p$, $c \in \mathbf{Z}$.

$$A_{\hat{q}}^n = \begin{pmatrix} \cos((ct+p)\theta) & -\alpha s_1 \sin((ct+p)\theta) & -\beta s_2 \sin((ct+p)\theta) & -\alpha\beta s_3 \sin((ct+p)\theta) \\ s_1 \sin((ct+p)\theta) & \cos((ct+p)\theta) & -\beta s_3 \sin((ct+p)\theta) & \beta s_2 \sin((ct+p)\theta) \\ s_2 \sin((ct+p)\theta) & \alpha s_3 \sin((ct+p)\theta) & \cos((ct+p)\theta) & -\alpha s_1 \sin((ct+p)\theta) \\ s_3 \sin((ct+p)\theta) & -s_2 \sin((ct+p)\theta) & s_1 \sin((ct+p)\theta) & \cos((ct+p)\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(p\theta) & -\alpha s_1 \sin(p\theta) & -\beta s_2 \sin(p\theta) & -\alpha\beta s_3 \sin(p\theta) \\ s_1 \sin(p\theta) & \cos(p\theta) & -\beta s_3 \sin(p\theta) & \beta s_2 \sin(p\theta) \\ s_2 \sin(p\theta) & \alpha s_3 \sin(p\theta) & \cos(p\theta) & -\alpha s_1 \sin(p\theta) \\ s_3 \sin(p\theta) & -s_2 \sin(p\theta) & s_1 \sin(p\theta) & \cos(p\theta) \end{pmatrix} = A_{\hat{q}}^{p}.$$

Now suppose

$$A_{\hat{q}}^{n} = \begin{pmatrix} \cos(n\theta) & -\alpha s_1 \sin(n\theta) & -\beta s_2 \sin(n\theta) & -\alpha\beta s_3 \sin(n\theta) \\ s_1 \sin(n\theta) & \cos(n\theta) & -\beta s_3 \sin(n\theta) & \beta s_2 \sin(n\theta) \\ s_2 \sin(n\theta) & \alpha s_3 \sin(n\theta) & \cos(n\theta) & -\alpha s_1 \sin(n\theta) \\ s_3 \sin(n\theta) & -s_2 \sin(n\theta) & s_1 \sin(n\theta) & \cos(n\theta) \end{pmatrix}$$

and

$$A_{\hat{q}}^{p} = \begin{pmatrix} \cos(p\theta) & -\alpha s_1 \sin(p\theta) & -\beta s_2 \sin(p\theta) & -\alpha\beta s_3 \sin(p\theta) \\ s_1 \sin(p\theta) & \cos(p\theta) & -\beta s_3 \sin(p\theta) & \beta s_2 \sin(p\theta) \\ s_2 \sin(p\theta) & \alpha s_3 \sin(p\theta) & \cos(p\theta) & -\alpha s_1 \sin(p\theta) \\ s_3 \sin(p\theta) & -s_2 \sin(p\theta) & s_1 \sin(p\theta) & \cos(p\theta) \end{pmatrix}.$$

Since $A_{\hat{q}}^{n} = A_{\hat{q}}^{p}$, $\sin n\theta = \sin p\theta$, and $\cos n\theta = \cos p\theta$, $n\theta = p\theta + 2\pi c$, $c \in \mathbf{Z}$; thus, $n = c\dfrac{2\pi}{\theta} + p$, $n = p(\mathrm{mod} \quad t)$.

Example 4.2

Let $\hat{q} = \dfrac{\sqrt{2}}{2} + \dfrac{1}{2\sqrt{\alpha}} i + \dfrac{1}{2\sqrt{\beta}} j + 2\varepsilon k$ as a unit dual generalized quaternion, the polar form $\hat{q} = \cos\dfrac{\pi}{4} + \vec{s}\sin\dfrac{\pi}{4}$, where $\vec{s} = \dfrac{1}{\sqrt{2\alpha}} i + \dfrac{1}{\sqrt{2\beta}} j + 2\sqrt{2}\varepsilon k = (s_1, s_2, s_3)$. From **Theorem 4.1** and **Theorem 4.2**, we have

$$A_{\hat{q}} = A_{\hat{q}}^{9} = A_{\hat{q}}^{17} = A_{\hat{q}}^{25} = \cdots \qquad A_{\hat{q}}^{2} = A_{\hat{q}}^{10} = A_{\hat{q}}^{18} = A_{\hat{q}}^{26} = \cdots$$

$$A_{\hat{q}}^{3} = A_{\hat{q}}^{11} = A_{\hat{q}}^{19} = A_{\hat{q}}^{27} = \cdots \qquad A_{\hat{q}}^{4} = A_{\hat{q}}^{12} = A_{\hat{q}}^{20} = A_{\hat{q}}^{28} = \cdots = -\boldsymbol{I}_4$$

$$\ldots$$

$$A_{\hat{q}}^{8} = A_{\hat{q}}^{16} = A_{\hat{q}}^{24} = A_{\hat{q}}^{32} = \cdots = \boldsymbol{I}_4.$$

(2) **Case 2**

Assume α is a positive number and β is a negative number.

For a dual generalized quaternion $\hat{q} = \hat{q}_0 + \hat{q}_1 i + \hat{q}_2 j + \hat{q}_3 k$, De Moivre's theorem of $A_{\hat{q}}$ is discussed in three cases, respectively.

Case A

When the norm of dual generalized quaternion is positive, and the norm of its vector part is negative, $N_{\hat{q}} = \hat{q}_0^2 + \alpha\hat{q}_1^2 + \beta\hat{q}_2^2 + \alpha\beta\hat{q}_3^2 > 0$ and $N_{\vec{V}_{\hat{q}}} = \alpha\hat{q}_1^2 + \beta\hat{q}_2^2 + \alpha\beta\hat{q}_3^2 < 0$. In this case, the polar form is written as $\hat{q} = r\left(\cosh\hat{\theta} + \vec{\eta}\sinh\hat{\theta}\right)$ [18], where $r = \sqrt{N_{\hat{q}}}$, $\cosh\hat{\theta} = \frac{\hat{q}_0}{r}$, $\sinh\hat{\theta} = \frac{\sqrt{-\alpha\hat{q}_1^2 - \beta\hat{q}_2^2 - \alpha\beta\hat{q}_3^2}}{r}$, $\hat{\theta} = \theta + \varepsilon\theta^*$, and the unit dual vector $\vec{\eta}$ is defined as $\vec{\eta} = \frac{\hat{q}_1 i + \hat{q}_2 j + \hat{q}_3 k}{\sqrt{-\alpha\hat{q}_1^2 - \beta\hat{q}_2^2 - \alpha\beta\hat{q}_3^2}} = (\eta_1, \eta_2, \eta_3)$, $\vec{\eta}^2 = 1$.

Theorem 4.3

Let $\hat{q} = r\left(cosh\hat{\theta} + \vec{\eta}sinh\hat{\theta}\right)$ as a dual generalized quaternion, $N_{\hat{q}} > 0$ and $N_{\vec{V}_{\hat{q}}} < 0$. For integer t,

$$A_{\hat{q}}^t = r^t \begin{pmatrix} cosh(t\hat{\theta}) & -\alpha\eta_1 sinh(t\hat{\theta}) & -\beta\eta_2 sinh(t\hat{\theta}) & -\alpha\beta\eta_3 sinh(t\hat{\theta}) \\ \eta_1 sinh(t\hat{\theta}) & cosh(t\hat{\theta}) & -\beta\eta_3 sinh(t\hat{\theta}) & \beta\eta_2 sinh(t\hat{\theta}) \\ \eta_2 sinh(t\hat{\theta}) & \alpha\eta_3 sinh(t\hat{\theta}) & cosh(t\hat{\theta}) & -\alpha\eta_1 sinh(t\hat{\theta}) \\ \eta_3 sinh(t\hat{\theta}) & -\eta_2 sinh(t\hat{\theta}) & \eta_1 sinh(t\hat{\theta}) & cosh(t\hat{\theta}) \end{pmatrix}.$$

Proof

Using induction on positive integers t. Assume that

$$A_{\hat{q}}^t = r^t \begin{pmatrix} cosh(t\hat{\theta}) & -\alpha\eta_1 sinh(t\hat{\theta}) & -\beta\eta_2 sinh(t\hat{\theta}) & -\alpha\beta\eta_3 sinh(t\hat{\theta}) \\ \eta_1 sinh(t\hat{\theta}) & cosh(t\hat{\theta}) & -\beta\eta_3 sinh(t\hat{\theta}) & \beta\eta_2 sinh(t\hat{\theta}) \\ \eta_2 sinh(t\hat{\theta}) & \alpha\eta_3 sinh(t\hat{\theta}) & cosh(t\hat{\theta}) & -\alpha\eta_1 sinh(t\hat{\theta}) \\ \eta_3 sinh(t\hat{\theta}) & -\eta_2 sinh(t\hat{\theta}) & \eta_1 sinh(t\hat{\theta}) & cosh(t\hat{\theta}) \end{pmatrix}$$

holds. Then,

$$A_{\hat{q}}^{t+1} = r^{t+1} \begin{pmatrix} cosh((t+1)\hat{\theta}) & -\alpha\eta_1 sinh((t+1)\hat{\theta}) & -\beta\eta_2 sinh((t+1)\hat{\theta}) & -\alpha\beta\eta_3 sinh((t+1)\hat{\theta}) \\ \eta_1 sinh((t+1)\hat{\theta}) & cosh((t+1)\hat{\theta}) & -\beta\eta_3 sinh((t+1)\hat{\theta}) & \beta\eta_2 sinh((t+1)\hat{\theta}) \\ \eta_2 sinh((t+1)\hat{\theta}) & \alpha\eta_3 sinh((t+1)\hat{\theta}) & cosh((t+1)\hat{\theta}) & -\alpha\eta_1 sinh((t+1)\hat{\theta}) \\ \eta_3 sinh((t+1)\hat{\theta}) & -\eta_2 sinh((t+1)\hat{\theta}) & \eta_1 sinh((t+1)\hat{\theta}) & cosh((t+1)\hat{\theta}) \end{pmatrix},$$

Hence, the formula is true. Moreover, since

$$A_{\hat{q}}^{-1} = r^{-1}\begin{pmatrix} cosh\hat{\theta} & \alpha\eta_1 sinh\hat{\theta} & \beta\eta_2 sinh\hat{\theta} & \alpha\beta\eta_3 sinh\hat{\theta} \\ -\eta_1 sinh\hat{\theta} & cosh\hat{\theta} & \beta\eta_3 sinh\hat{\theta} & -\beta\eta_2 sinh\hat{\theta} \\ -\eta_2 sinh\hat{\theta} & -\alpha\eta_3 sinh\hat{\theta} & cosh\hat{\theta} & \alpha\eta_1 sinh\hat{\theta} \\ -\eta_3 sinh\hat{\theta} & \eta_2 sinh\hat{\theta} & -\eta_1 sinh\hat{\theta} & cosh\hat{\theta} \end{pmatrix},$$

$$A_{\hat{q}}^{-t} = r^{-t}\begin{pmatrix} cosh\left(-t\hat{\theta}\right) & -\alpha\eta_1 sinh\left(-t\hat{\theta}\right) & -\beta\eta_2 sinh\left(-t\hat{\theta}\right) & -\alpha\beta\eta_3 sinh\left(-t\hat{\theta}\right) \\ \eta_1 sinh\left(-t\hat{\theta}\right) & cosh\left(-t\hat{\theta}\right) & -\beta\eta_3 sinh\left(-t\hat{\theta}\right) & \beta\eta_2 sinh\left(-t\hat{\theta}\right) \\ \eta_2 sinh\left(-t\hat{\theta}\right) & \alpha\eta_3 sinh\left(-t\hat{\theta}\right) & cosh\left(-t\hat{\theta}\right) & -\alpha\eta_1 sinh\left(-t\hat{\theta}\right) \\ \eta_3 sinh\left(-t\hat{\theta}\right) & -\eta_2 sinh\left(-t\hat{\theta}\right) & \eta_1 sinh\left(-t\hat{\theta}\right) & cosh\left(-t\hat{\theta}\right) \end{pmatrix}$$

$$= r^{-n}\begin{pmatrix} cosh\left(t\hat{\theta}\right) & \alpha\eta_1 sinh\left(t\hat{\theta}\right) & \beta\eta_2 sinh\left(t\hat{\theta}\right) & \alpha\beta\eta_3 sinh\left(t\hat{\theta}\right) \\ -\eta_1 sinh\left(t\hat{\theta}\right) & cosh\left(t\hat{\theta}\right) & \beta\eta_3 sinh\left(t\hat{\theta}\right) & -\beta\eta_2 sinh\left(t\hat{\theta}\right) \\ -\eta_2 sinh\left(t\hat{\theta}\right) & -\alpha\eta_3 sinh\left(t\hat{\theta}\right) & cosh\left(t\hat{\theta}\right) & \alpha\eta_1 sinh\left(t\hat{\theta}\right) \\ -\eta_3 sinh\left(t\hat{\theta}\right) & \eta_2 sinh\left(t\hat{\theta}\right) & -\eta_1 sinh\left(t\hat{\theta}\right) & cosh\left(t\hat{\theta}\right) \end{pmatrix}.$$

Then, the formula holds for every integer.

Example 4.3

Let $\hat{q} = \sqrt{2} + \frac{1}{\sqrt{2}} i + j + \varepsilon k \in \hat{\mathbb{H}}_{\alpha\beta}$. Consider the special case, i.e., if $\alpha = 1$, $\beta = -1$, the polar form is $\hat{q} = \frac{\sqrt{6}}{2}\left(\frac{2}{\sqrt{3}} + \vec{\eta}\,\frac{1}{\sqrt{3}}\right)$, where $\vec{\eta} = i + \sqrt{2} j + \varepsilon\sqrt{2} k = \left(\eta_1, \eta_2, \eta_3\right)$. All powers of the matrix $A_{\hat{q}}$ are found based on the **Theorem 4.3**. For example, the 2-nd and 4-th powers are

$$A_{\hat{q}}^2 = \begin{pmatrix} \frac{5}{2} & -2 & 2\sqrt{2} & 2\sqrt{2}\varepsilon \\ 2 & \frac{5}{2} & 2\sqrt{2}\varepsilon & -2\sqrt{2} \\ 2\sqrt{2} & 2\sqrt{2}\varepsilon & \frac{5}{2} & -2 \\ 2\sqrt{2}\varepsilon & -2\sqrt{2} & 2 & \frac{5}{2} \end{pmatrix}, \quad A_{\hat{q}}^4 = \begin{pmatrix} \frac{41}{4} & -10 & 10\sqrt{2} & 10\sqrt{2}\varepsilon \\ 10 & \frac{41}{4} & 10\sqrt{2}\varepsilon & -10\sqrt{2} \\ 10\sqrt{2} & 10\sqrt{2}\varepsilon & \frac{41}{4} & -10 \\ 10\sqrt{2}\varepsilon & -10\sqrt{2} & 10 & \frac{41}{4} \end{pmatrix}.$$

Case B

When the norm of dual generalized quaternion is positive and the norm of its vector part is to be positive, $N_{\hat{q}} > 0$ and $N_{\vec{V}_{\hat{q}}} = \alpha\hat{q}_1^2 + \beta\hat{q}_2^2 + \alpha\beta\hat{q}_3^2 > 0$. In this case, the polar form is $\hat{q} = r\left(\cos\hat{\theta} + \vec{\omega}\sin\hat{\theta}\right)$ [18], where $r = \sqrt{N_{\hat{q}}}$, $\cos\hat{\theta} = \frac{\hat{q}_0}{r}$, $\sin\hat{\theta} = \frac{\sqrt{\alpha\hat{q}_1^2 + \beta\hat{q}_2^2 + \alpha\beta\hat{q}_3^2}}{r}$, $\hat{\theta} = \theta + \varepsilon\theta^*$, and the unit dual vector is defined as $\vec{\omega} = \frac{\hat{q}_1 i + \hat{q}_2 j + \hat{q}_3 k}{\sqrt{\alpha\hat{q}_1^2 + \beta\hat{q}_2^2 + \alpha\beta\hat{q}_3^2}} = (\omega_1, \omega_2, \omega_3)$, $\vec{\omega}^2 = -1$.

Theorem 4.4

Assume $\hat{q} = r\left(\cos\hat{\theta} + \vec{\omega}\sin\hat{\theta}\right)$ is a dual generalized quaternion, $N_{\hat{q}} > 0$ and $N_{\vec{V}_{\hat{q}}} > 0$. For any integer n,

$$A_{\hat{q}}^n = r^n \begin{pmatrix} \cos(n\hat{\theta}) & -\alpha\omega_1\sin(n\hat{\theta}) & -\beta\omega_2\sin(n\hat{\theta}) & -\alpha\beta\omega_3\sin(n\hat{\theta}) \\ \omega_1\sin(n\hat{\theta}) & \cos(n\hat{\theta}) & -\beta\omega_3\sin(n\hat{\theta}) & \beta\omega_2\sin(n\hat{\theta}) \\ \omega_2\sin(n\hat{\theta}) & \alpha\omega_3\sin(n\hat{\theta}) & \cos(n\hat{\theta}) & -\alpha\omega_1\sin(n\hat{\theta}) \\ \omega_3\sin(n\hat{\theta}) & -\omega_2\sin(n\hat{\theta}) & \omega_1\sin(n\hat{\theta}) & \cos(n\hat{\theta}) \end{pmatrix}.$$

Proof

Using the induction, the theorem can be verified as true.

Example 4.4

Let $\hat{q} = 1 + \sqrt{3}i + \varepsilon j - \varepsilon k \in \widehat{\mathbb{H}}_{\alpha\beta}$, if $\alpha = 1$ and $\beta = -1$, the polar form is $\hat{q} = 2\left(\frac{1}{2} + \vec{\omega}\frac{\sqrt{3}}{2}\right)$, where $\vec{\omega} = i + \frac{\varepsilon}{\sqrt{3}} j - \frac{\varepsilon}{\sqrt{3}} k = (\omega_1, \omega_2, \omega_3)$. From **Theorem 4.4**, any power of the matrix $A_{\hat{q}}$ can be obtained, for example, the 3-rd and 6-th powers are $A_{\hat{q}}^3 = -8\boldsymbol{I}_4$ and $A_{\hat{q}}^6 = 64\boldsymbol{I}_4$, respectively.

Case C

When the norm of the dual generalized quaternion is negative, $N_{\hat{q}} = \hat{q}_0^2 + \alpha\hat{q}_1^2 + \beta\hat{q}_2^2 + \alpha\beta\hat{q}_3^2 < 0$.

Since $0 < \hat{q}_0^2 < -\alpha\hat{q}_1^2 - \beta\hat{q}_2^2 - \alpha\beta\hat{q}_3^2$, thus $\alpha\hat{q}_1^2 + \beta\hat{q}_2^2 + \alpha\beta\hat{q}_3^2 < 0$, i.e., $N_{\vec{V}_{\hat{q}}} < 0$. In this case, the polar form is $\hat{q} = r\left(\sinh\hat{\theta} + \vec{\psi}\cosh\hat{\theta}\right)$ [18], where $r = \sqrt{\left|N_{\hat{q}}\right|}$, $\cosh\hat{\theta} = \frac{\sqrt{-\alpha\hat{q}_1^2 - \beta\hat{q}_2^2 - \alpha\beta\hat{q}_3^2}}{r}$, $\sinh\hat{\theta} = \frac{\hat{q}_0}{r}$, $\hat{\theta} = \theta + \varepsilon\theta^*$, and the unit dual vector $\vec{\psi}$ is defined as $\vec{\psi} = \frac{\hat{q}_1 i + \hat{q}_2 j + \hat{q}_3 k}{\sqrt{-\alpha\hat{q}_1^2 - \beta\hat{q}_2^2 - \alpha\beta\hat{q}_3^2}} = (\psi_1, \psi_2, \psi_3)$, $\vec{\psi}^2 = 1$.

Theorem 4.5

Assuming $\hat{q} = r\left(sinh\hat{\theta} + \vec{\psi} cosh\hat{\theta}\right)$ is a dual generalized quaternion and $N_{\hat{q}} < 0$, then, when t is an odd number,

$$A_{\hat{q}}^{t} = r^{t}\begin{pmatrix} sinh(t\hat{\theta}) & -\alpha\psi_1 cosh(t\hat{\theta}) & -\beta\psi_2 cosh(t\hat{\theta}) & -\alpha\beta\psi_3 cosh(t\hat{\theta}) \\ \psi_1 cosh(t\hat{\theta}) & sinh(t\hat{\theta}) & -\beta\psi_3 cosh(t\hat{\theta}) & \beta\psi_2 cosh(t\hat{\theta}) \\ \psi_2 cosh(t\hat{\theta}) & \alpha\psi_3 cosh(t\hat{\theta}) & sinh(t\hat{\theta}) & -\alpha\psi_1 cosh(t\hat{\theta}) \\ \psi_3 cosh(t\hat{\theta}) & -\psi_2 cosh(t\hat{\theta}) & \psi_1 cosh(t\hat{\theta}) & sinh(t\hat{\theta}) \end{pmatrix};$$

when t is an even number,

$$A_{\hat{q}}^{t} = r^{t}\begin{pmatrix} cosh(t\hat{\theta}) & -\alpha\psi_1 sinh(t\hat{\theta}) & -\beta\psi_2 sinh(t\hat{\theta}) & -\alpha\beta\psi_3 sinh(t\hat{\theta}) \\ \psi_1 sinh(t\hat{\theta}) & cosh(t\hat{\theta}) & -\beta\psi_3 sinh(t\hat{\theta}) & \beta\psi_2 sinh(t\hat{\theta}) \\ \psi_2 sinh(t\hat{\theta}) & \alpha\psi_3 sinh(t\hat{\theta}) & cosh(t\hat{\theta}) & -\alpha\psi_1 sinh(t\hat{\theta}) \\ \psi_3 sinh(t\hat{\theta}) & -\psi_2 sinh(t\hat{\theta}) & \psi_1 sinh(t\hat{\theta}) & cosh(t\hat{\theta}) \end{pmatrix}.$$

Proof

Using the induction, the theorem can be verified as true.

Example 4.5

Let $\hat{q} = \varepsilon + \varepsilon i - j - k \in \tilde{\mathbb{H}}_{\alpha\beta}$, if $\alpha = 1$, $\beta = -1$, the polar form of $\hat{q} = \sqrt{2}\left(\frac{\varepsilon}{\sqrt{2}} + \vec{\psi}\cdot 1\right)$, where $\vec{\psi} = \frac{\sqrt{2}\varepsilon}{2} i - \frac{\sqrt{2}}{2} j - \frac{\sqrt{2}}{2} k = (\psi_1, \psi_2, \psi_3)$. All powers for the matrix $A_{\hat{q}}$ are found with the aid of **Theorem 4.5**.

If $n = 2t$, then

$$A_{\hat{q}}^{2t} = 2^{t}\begin{pmatrix} 1 & 0 & -t\varepsilon & -t\varepsilon \\ 0 & 1 & -t\varepsilon & t\varepsilon \\ -t\varepsilon & -t\varepsilon & 1 & 0 \\ -t\varepsilon & t\varepsilon & 0 & 1 \end{pmatrix};$$

If $n=2t+1$, then

$$A_{\hat{q}}^{2t+1}=2^{t}\begin{pmatrix}(2t+1)\varepsilon & -\varepsilon & -1 & -1\\ \varepsilon & (2t+1)\varepsilon & -1 & 1\\ -1 & -1 & (2t+1)\varepsilon & -\varepsilon\\ -1 & 1 & \varepsilon & (2t+1)\varepsilon\end{pmatrix},$$

where $t\in\mathbf{Z}$.

5. Conclusion

In this paper, the matrix representation for dual generalized quaternions was obtained, and De Moivre's theorem and Euler's formula were investigated for these matrices. Additionally, the relations between the powers of matrices associated with dual generalized quaternions were also obtained.

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