

A Study on the Teaching of Mathematical Methods for Physics based on the Laws of Scientific Cognition: The Introduction of Complex Numbers as an Example

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Abstract: The development of innovative talent should be guided by the principles of scientific cognition. Currently, there is a significant gap in students' understanding of the origin, visualization, and necessity of complex numbers, particularly in their application to physics. This paper examines the historical development of complex numbers, tracing contributions from Ferro, Cardano, and Bombelli to Gauss's systematic geometric representation. It emphasizes the necessity and geometric significance of introducing complex numbers. Additionally, the paper explores the critical role of complex numbers in physics, especially in quantum mechanics. Using the Schrödinger equation as a case study, we demonstrate that the introduction of complex numbers not only ensures the existence of solutions but also provides a natural framework for describing the phase evolution and probability amplitudes of wave functions for microscopic particles. Through a thorough analysis of the motivations for the introduction and application of complex numbers, this paper aims to enhance students' understanding of their mathematical nature, geometric visualization, and physical significance, ultimately contributing to the development of innovative talent.

Keywords: Complex numbers; Visualization of complex numbers; Schrödinger equation; Scientific cognition principles

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1. Introduction

The section on complex function theory in Mathematical Methods for Physics is often regarded as one of the most elegant mathematical developments of the 19th century^[1], with its first section focusing on complex numbers. However, the author observes that most textbooks on mathematical physics methods typically introduce the concept of complex numbers and their operations directly, with few addressing the historical context or the necessity of introducing complex numbers. While college students generally encounter complex numbers in high school, learning to perform basic operations such as addition, subtraction, multiplication, and division, the author

notes that many students lack a deeper understanding of why complex numbers were introduced, how they are visualized, and their necessity in physics. Furthermore, some students perceive complex numbers as abstract constructs invented by mathematicians, failing to recognize their real existence. Many are also unaware that the introduction of complex numbers was not to make quadratic equations solvable, but rather that the root formula for quadratic equations led to the need for complex numbers.

The cultivation of innovative talent must be grounded in the principles of scientific cognition. Drawing on teaching experience, the author applies the law of scientific cognition to explain the introduction of complex numbers, addressing how the solution to is derived and demonstrating the historical and physical necessity of complex numbers ^[2]. This approach aims to help students recognize that complex numbers are indeed real numbers, that the solution to is valid, and that complex numbers are essential in physics. By doing so, students will gain a deeper understanding of the scientific inquiry process.

In the early days of mathematics, solving quadratic equations was problematic, as negative numbers had not yet been introduced. For example, equations like, were unsolvable, and the concept of negative values was not easily understood. However, as new concepts such as debt and temperature emerged, negative numbers gradually gained acceptance ^[3]. Humans have long known the root formula for quadratic equations, and it has been demonstrated that the Babylonians were proficient at solving quadratic equations ^[4]. However, for the equation , the root formula produces , and people at the time could not comprehend the notion of negative values under the square root. Consequently, they concluded that no solution existed for this equation. Graphically, the function does not intersect the x-axis, making the absence of solutions intuitive.



Figure 1. Historical figures associated with the formula for finding the roots of a cubic equation in one degree.

The situation changed drastically when mathematicians turned to solving cubic equations (**Figure 1**). Unlike in ancient Babylon, where quadratic equations were solved over 4,000 years ago, it was not until around 1500 that Scipione del Ferro (1465–1526) of the University of Bologna discovered a method for solving cubic equations. Ferro discovered the method for deriving the formula to find the roots of a cubic equation, but kept his solution secret until his death, never revealing the formula publicly. The primary reason for this secrecy stemmed from the intellectual climate of his time. Unlike today’s universities or research institutions, which derive academic prestige from the publication of scholarly papers, individuals in the Renaissance period earned their reputations through public problem-solving contests, often gaining recognition by challenging one another

in public duels of intellect.

Before his death, Ferro passed the formula for solving quadratic equations to his pupil Fior and his son-in-law, Nave. In 1535, the ambitious and impetuous Fior approached Tartaglia, one of the most prominent mathematicians of the era, seeking a mathematical duel. However, Tartaglia had already solved the quadratic equation problem, leaving Fior's challenge unsuccessful. It was in this same year that Fior, unaware of Tartaglia's solution to the cubic equation, requested a duel over the cubic problem. To his surprise, Tartaglia had also derived a solution for the cubic equation. Later, Cardano, a mathematician of note, sought Tartaglia's formula for solving cubic equations and persuaded him to share it under the condition of secrecy. However, by chance, Cardano discovered that Ferro's formula was identical to Tartaglia's during a visit to Ferro's son-in-law, Nave. Realizing that the secret no longer needed to be kept, Cardano published the formula for finding the roots of a cubic equation in his influential work *Ars Magna* (1545). The root of a cubic equation (1) can be solved using the formula (2) [5]:

$$x^3 + cx = d \quad (1)$$

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} \quad (2)$$

This rooting formula is much more complicated than the rooting formula for quadratic equations and can solve some of the quadratic equations, but for the equation $x^3 - 15x = 4$, the solution using the rooting formula is:

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \quad (3)$$

The result will have $\sqrt{-121}$ in it, and according to conventional wisdom, once there is a negative number under the root sign, the equation is unsolvable.

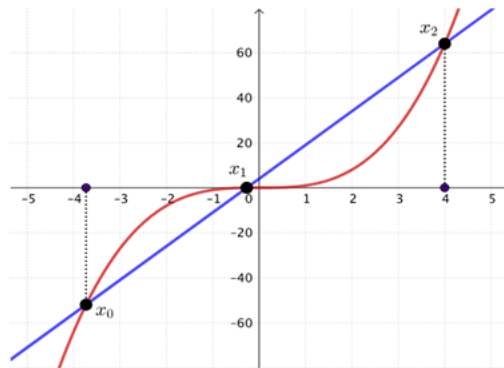


Figure 2. Image of the function of x^3 and $15x + 4$.

However, if we approach it from a different perspective and move to the right-hand side of the equation, we then have two equations: $y = x^3$ and $y = 15x + 4$ as shown in **Figure 2**. Since x^3 grows faster than the linear function, it is evident that, for $x > 0$, these two functions must intersect at some point within this range. By using a trial-and-error method, we can determine that $x = 4$ is a solution. Initially, applying the root formula suggested there was no solution, but through this alternative approach, we arrive at a valid solution. This creates a contradiction, which, as the history of science has repeatedly shown, often signals a breakthrough—

contradictions are frequently where scientific progress is made.

To resolve this issue, Bombelli introduced the symbol $\sqrt{-1}$ in 1572 ^[6], and we define:

$$\sqrt[3]{2 + \sqrt{-121}} = a + b\sqrt{-1} \quad (4)$$

$$\sqrt[3]{2 - \sqrt{-121}} = a - b\sqrt{-1} \quad (5)$$

Taking the cube of both sides and simplifying, we obtain:

$$2 = a(a^2 - 3b^2) \quad (6)$$

$$11 = b(3a^2 - b^2) \quad (7)$$

Solving equations (6) and (7) gives $a = 2$ and $b = -1$. Therefore,

$$x = 2 + \sqrt{-1} + 2 - \sqrt{-1} \quad (8)$$

The two $\sqrt{-1}$ in the result cancel out exactly, leading to the solution $x = 4$ ^[7], which is an exciting result because one can actually calculate $x = 4$ from a bunch of complicated equations, and although Bombelli himself considers it a sophistry, his method can inspire others.

Later, Descartes coined the term “imaginary number” for expressions like $\sqrt{-1}$ ^[8,9], which translates as “虚数” in Chinese, reflecting the prevailing view among the most prominent mathematicians of the time that complex numbers were not considered “real” numbers. In 1777, Euler adopted the letter “i” from the word “imaginary” to represent $\sqrt{-1}$ ^[10]. Despite being one of the most brilliant mathematicians of his era, Euler’s understanding of complex numbers was not entirely complete. This is evident from his work, where we find the following equation:

$$\sqrt{-2} \times \sqrt{-3} = \sqrt{6} \quad (9)$$

Obviously, this is wrong, it is known that the correct result for $\sqrt{-2} \times \sqrt{-3}$ is $-\sqrt{6}$, but Euler got the correct Euler’s formula ^[11] without being clear about the geometrical meaning of imaginary units.

It was not until later, in 1797, that the Norwegian surveyor Caspar Wessel and the French mathematician Argand (in 1806) first proposed representing complex numbers using a geometric plane, which we now refer to as the complex plane ^[12]. The complex plane consists of a real axis and an imaginary axis perpendicular to it. It is important to note that this plane is not simply a repetition of the Cartesian coordinate system. As we know, $i^2 = -1$, which leads to the following:

$$1 \times i \times i = -1 \quad (10)$$

This implies that 1 becomes -1 after two multiplications by i , indicating that each multiplication by i corresponds to a 90° rotation, as illustrated in **Figure 3**. Therefore, the geometric interpretation of i is that the imaginary unit i is positioned one unit away from the origin along the imaginary axis. Any number multiplied by i represents a 90° rotation. The imaginary number is essentially a real number, but it exists on a line perpendicular to the real axis, in a different dimension. This provides a natural foundation for the concept of the complex plane, as shown in **Figure 4**.

The concept here suggests that complex numbers are, in fact, real numbers, but with dimensions. A complex

number can be considered a two-dimensional number, with its second dimension lying along a direction perpendicular to the familiar real axis. It is essential to think of a complex number as an extension of real numbers. In fact, numbers exist not only in two dimensions but also in higher dimensions, such as four, eight, sixteen, and beyond. However, for the purposes of physics, the domain of complex numbers is sufficiently expansive. With a clear understanding of the geometric significance of the imaginary unit, the development of complex numbers took a more structured path. This was further advanced through the work of mathematicians such as Gauss, Cauchy, and Riemann, eventually leading to the modern theory of functions of a complex variable. It took 250 years from the discovery of the cubic root formula to the explanation of the complex plane. During this period, the understanding of complex numbers evolved in a spiral, progressing in a complex and winding manner. As Marx once said, there are no smooth highways in science; only those who are undaunted by hardship and climb the steep mountain paths can reach the brilliant summit.

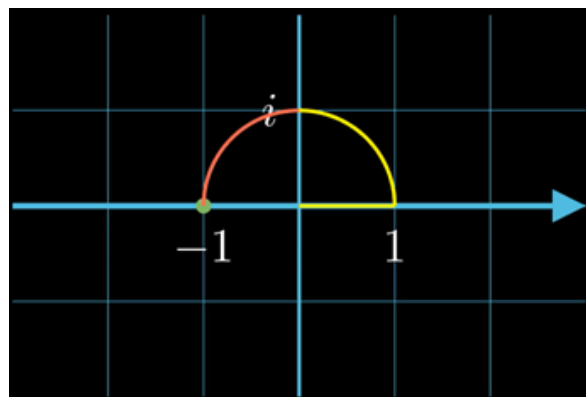


Figure 3. Geometric meaning and representation of imaginary numbers.

The visualization of complex numbers presents a more challenging task. For the function $f(z) = z^2 + 1$, both the independent variable and the function value are complex numbers. Here, we represent the independent variable as $z = x + yi$, and the function value as $f(x) = u + vi$. This results in four variables, requiring a four-dimensional space to represent the function's image, which exceeds typical understanding. However, we can still find a solution. By using a false-color map, we can represent the three variables x , y , and u , while the fourth variable, v , is depicted using different colors^[13]. With the aid of MATLAB software, the function image is shown in **Figure 5**. In two-dimensional space, the function $f(z) = z^2 + 1$ indeed has two intersections with $u = 0$, which occur at $\pm i$.

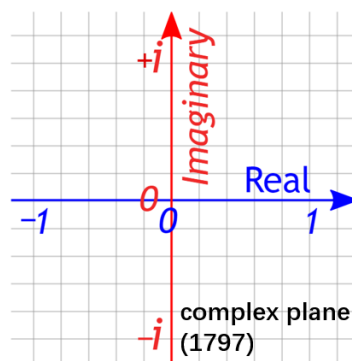


Figure 4. Complex plane representation of complex numbers.

At the same time, complex numbers are essential in physics as shown in **Figure 6** ^[14]. In Schrödinger's equation in quantum mechanics, i appears, which means that in order for the Schrödinger equation to have a solution, the wave function describing microscopic particles must be a complex number. The introduction of complex numbers not only ensures the existence of a solution to the equation but also provides a natural description of the phase evolution and probability amplitude of the wave function of microscopic particles. As Freeman Dyson stated, the $\sqrt{-1}$ in Schrödinger's equation signifies that nature operates through complex numbers, rather than real numbers ^[15].

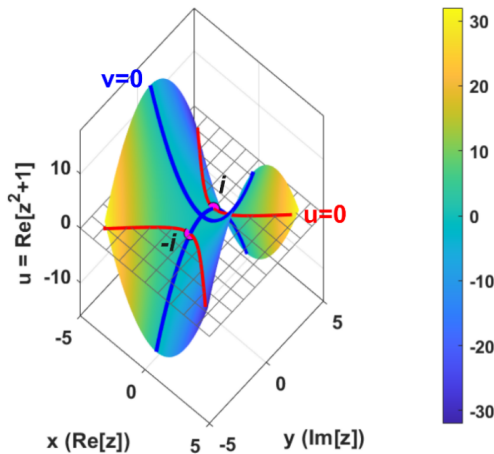


Figure 5. Visualization image of the roots of the equation .

Teaching functions of a complex variable in a mathematical physics methods course offers students a deeper understanding of the nature and significance of complex numbers by integrating both historical context and practical applications. The introduction of complex numbers originated from the necessity of solving cubic equations, which in turn led to the development of imaginary numbers and the complex plane. By conceptualizing complex numbers as two-dimensional quantities and visualizing them through geometric representations and rotational interpretations, students can better appreciate their real-world relevance and importance. In physics, the essential role of complex numbers in quantum mechanics further underscores their indispensability in describing microscopic phenomena. This teaching approach, grounded in the principles of scientific cognition, not only helps students acquire theoretical knowledge but also fosters their scientific literacy and enhances their ability to think innovatively.

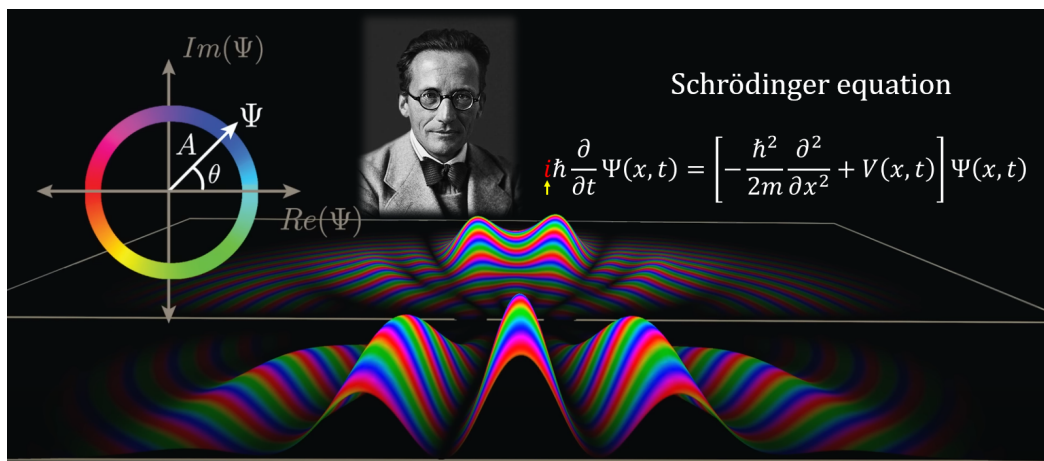


Figure 6. Schrödinger and the Schrödinger equation.

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Disclosure statement

The author declares no conflict of interest.

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