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The Time-Consistent Optimal Reinsurance Strategy of Insurance Group under the CEV Model

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Abstract: The article introduces proportional reinsurance contracts under the mean-variance criterion, studying the time-consistence investment portfolio problem considering the interests of both insurance companies and reinsurance companies. The insurance claims process follows a jump-diffusion model, assuming that the risk asset prices of insurance companies and reinsurance companies follow CEV models different from each other. In the framework of game theory, the time-consistent equilibrium reinsurance strategy is obtained by solving the extended HJB equation analytically. Finally, numerical examples are used to illustrate the impact of model parameters on equilibrium strategies and provide economic explanations. The results indicate that the decision weights of insurance companies and reinsurance companies do have a significant impact on both the reinsurance ratio and the equilibrium reinsurance strategy.

Keywords: Mean-variance; Joint benefit; Extended HJB equation; Constant elastic variance model

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1. Introduction

Insurance companies can appropriately use reinsurance to share the risks borne by insurers. Reinsurance is also an effective mechanism for risk sharing within insurance groups, such as tax reduction and increased profitability. The mean-variance criterion is one of the criteria for researching reinsurance investment strategies. Its main advantage is that it considers both safety and profitability, achieving a balance between terminal returns and risks [1-3]. However, because the variance lacks iterative, dynamic mean-variance is time-inconsistent. Time inconsistency can be resolved by a pre-commitment strategy, namely establishing an optimal strategy at the initial time and using this strategy at every moment in the future. However, this strategy is not optimal for a future moment [4,5]. Another solution is to provide investors with a time-consistent strategy: Björk and Murgoci studied generally controlled Markov processes and target functionals within the framework of game theory, obtaining an extended Hamilton-Jacobi-Bellman (HJB) equation, using the form of nonlinear system equations to determine equilibrium strategies and equilibrium value functions [6]. Zeng and Li first provided a general verification theorem of the Black-Scholes model within the Nash equilibrium framework of Björk and Murgoci and derived the explicit solution of the optimal time-consistent strategy and optimal value function of reinsurance investment [7]. Zhu *et al.* studied the nonzero and random differential games between two insurance companies under the mean-variance criterion and established extended HJB equations for the situations before and after default, providing closed-form solutions for Nash equilibrium in insurance and investment strategies [8]. In recent years, an increasing number of scholars have focused on the investment-reinsurance problem of insurance companies under stochastic interest rate models, such as using models like the constant elasticity of variance (CEV) model and Heston model to replace constant interest rate geometric Brownian motion to describe risk assets. Scholars such as Nie [9], Gu [10], and Cai [11] have considered the optimal investment and reinsurance problems of insurance companies under different utility functions when the price of risk assets follows the CEV model.

Although there have been fruitful research results on the investment-reinsurance strategies of insurance companies based on the CEV model, there are still the following shortcomings: Firstly. previous literature only explains the optimal investment strategy of insurance companies from the perspective of insurance companies, without considering that the optimal investment strategy of insurance companies may not necessarily be accepted by reinsurance companies, thus neglecting the profit issue of insurance groups (groups that have both insurance companies and reinsurance companies). Secondly, few scholars have considered the problem of time-consistent optimal strategy for insurance groups under the mean-variance criterion. This paper not only considers the optimal strategies of both insurance companies and reinsurance companies but also assumes that the claims process conforms to the jump-diffusion model, which is more in line with the reality of financial markets.

The remaining structure of this paper is as follows: In Section 2, the wealth process model of the insurance group is established; Section 3 presents a validation theorem for a general problem; Section 4 obtains a time-consistent investment-reinsurance equilibrium strategy by solving the extended HJB equation within the framework of game theory; Section 5 provides numerical analysis and economic explanations of the theoretical results; Section 6 is the conclusion of this paper; All proofs are included in Appendix A.

2. Money market

2.1. Wealth process

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $\{\mathcal{F}_t, t > 0\}$ be a flow defined on this probability space, which includes all P-sets and is right continuous. All stochastic processes in this paper are defined on this domain flow and are adaptive. Considering unforeseen events in the real world, we assume that the claims process of the insurance company follows a jump-diffusion model, as follows:

$$dC(t) = mdt - ndt + d\sum_{i=1}^{N(t)} Y_i \tag{1}$$

where m and n are two positive constants, W_0 is a standard Brownian motion. $N(t)$ is a homogeneous Poisson process with intensity $\lambda_0 > 0$, representing the number of claims occurring in the time interval $[0,T]$. Y_i, $i = 1,2,\ldots$ are independent and identically distributed (i,i,d) positive random variables, with mean $\mu_y = E[Y_i]$ and variance $\sigma_y^2 = E[Y_i^2]$.

Furthermore, assuming that the premium rate charged by the insurer is calculated based on the principle of expectation, then the premium rate is $c = (1+\theta)$, where $\theta > 0$ and θ is the safety load factor of the insurance company. The surplus process of the insurance company is determined as follows:

$$dR_1(t) = \theta mdt + ndW_0(t) - d\sum_{i=1}^{N(t)} Y_i \tag{2}$$

Here, θ_m =(1+ θ) $\lambda_0\mu_y$. We assume that the insurance company and the reinsurance company enter into a proportional reinsurance agreement, and let $p(t)$ denote the insurance coverage ratio of the insurance company at time t. At the same time, we assume that the reinsurance premium rate is also charged according to the principle of expectation, and the loading factor of the reinsurance company, η, is greater than the insurance company, θ, i.e., the premium rate of the reinsurance company is higher than that of the insurance company, otherwise there is arbitrage space. In this case, the insurance company should pay a portion of the premium to the reinsurance company at a rate of $p(t)(1+\eta)\lambda_0\mu_y$. When there is a proportional reinsurance agreement, the surplus processes of the insurance companies and reinsurance companies are:

$$dR_1(t) = (\theta - \eta p(t))mdt + (1 - p(t))ndW_0(t) - (1 - p(t))d\sum_{i=1}^{N(t)} Y_i \tag{3}$$

$$dR_2(t) = \eta p(t)mdt + p(t)ndW_0(t) - p(t)d\sum_{i=1}^{N(t)} Y_i \tag{4}$$

Without the loss of generality, in addition to the insurance business, it is assumed that both insurance companies and reinsurance companies can invest in a risk-free asset and a different risk asset from each other. The price process of risk-free assets is given by $dB(t)= rB(t)dt$, where $B(0)=B>0$. Here, $r>0$ represents the risk-free interest rate. The price process of the risk asset satisfies the CEV model, which was first proposed by Cox and Ross [12] and can capture potential fluctuations sensitivity more sensitively. Therefore, it has been used in the price models of financial assets [13-15]. The price processes of the insurance companies and the reinsurance companies are respectively:

$$dS_1(t) = S_1(t)\big(\mu_1 dt + \sigma_1 S_1 dW_1(t)\big), S_1(0) = S_1 > 0 \tag{5}$$

$$dS_2(t) = S_2(t)(\mu_2 dt + \sigma_2 S_2 dW_2(t)), S_2(0) = S_2 > 0 \tag{6}$$

where the expected return rate of the risk asset $\mu_i>r$, volatility $\sigma_i>r$, and similar to previous research [16,17]. For convenience, we refer to $S_1(t)$ and $S_2(t)$ as "risk asset 1" and "risk asset 2," respectively. When i=1,2, W_1 and W_2 are correlated random sources from financial assets to $S_1(t)$ and $S_2(t)$, and W_0,W_1,W_2 are mutually independent.

Let $\mu(t)=(\pi_1(t),\pi_2(t),p(t))$ be a trading strategy, which includes the reinsurance strategy as well as investment strategies of the insurance company and the reinsurance company, where $\pi_1(t)$ and $\pi_2(t)$ represent the amount invested in the risk asset $S_1(t)$ by the insurance company and the amount invested in the risk asset $S_2(t)$ by the reinsurance company at the time t. Given a trading strategy $\mu(t)$, the wealth processes of insurance companies and reinsurance companies are:

$$\begin{aligned} dX^{\mu}(t) &= \big[rX^{\mu}(t) + m\big(\theta - \eta p(t) + \pi_1(t)(\mu_1 - r)\big)dt\big] + \pi_1(t)\sigma_1 dW_1(t) \\ &\quad -(1 - p(t))d\sum_{i=1}^{N(t)} Y_i + n(1 - p(t))dW_0(t) \end{aligned} \tag{7}$$

$$\begin{aligned} dY^{\mu}(t) = &\ \big[rY^{\mu}(t) + m\big(\eta p(t) + \pi_2(t)(\mu_2 - r)\big)dt\big] + \pi_2(t)\sigma_2 dW_2(t) \\ &-(1 - p(t))d\sum_{i=1}^{N(t)} Y_i + np(t)dW_0(t) \end{aligned} \tag{8}$$

At the same time, we consider the insurance groups making decisions based on the following weighting process:

$$dZ^{\mu}(t) = \alpha dX^{\mu}(t) + \beta dY^{\mu}(t) \quad (9)$$

where α and β are two constant in the interval [0,1], and **Equations 6–8** generate the weighted process:

$$\begin{aligned} dZ^{\mu}(t) = [rZ^{\mu}(t) + m\alpha\theta - m\eta p(t)(\alpha-\beta) + \alpha\pi_1(t)(\mu_1 - r) + \beta\pi_2(t)(\mu_2 - r)]dt \\ + \alpha\pi_1(t)\sigma_1 dW_1(t) + [\alpha n - np(t)(\alpha-\beta)]dW_0(t) + \beta\pi_2(t)\sigma_2 dW_2(t) \\ - [\alpha - p(t)(\alpha-\beta)]d\sum_{i=1}^{N(t)} Y_i \end{aligned} \quad (10)$$

Note that in **Equations 2.1–2.9**, α,β can be explained in two different ways. Firstly, insurance companies and reinsurance companies may belong to the same group company. In this case, the group company owns 100% α of the insurance company and 100% β of the reinsurance company, which $Z^{\mu}(t)$ can be interpreted as the total surplus of the group company. When the shares of insurance companies and reinsurance companies held by the group company are enough to dominate the boards of the two companies, the investment-reinsurance strategy of the two companies is determined by the insurance group $Z^{\mu}(t)$, so the optimal investment reinsurance strategy should be based on the weighted sum process of the group company.

Secondly, if we set β=1–α in $Z^{\mu}(t)$, then the parameter balances the interests of the insurance companies and reinsurance companies in determining the optimal investment reinsurance strategy. While α and β are known as decision weights, they are not considered controlling variables in this paper because they measure the relative decision ability of the insurance companies and the reinsurance companies.

In the admissible strategy (**Definition 1**), for $t\in[t,T]$, a trading strategy is $\{\mu(v)=(\pi_1(v),\pi_2(v),p(v)) \cdot v\in[t,T]\}$ said to be admissible for initial condition $(t,z,s_1,s_2,r) \in [0,T]\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+$ if it satisfies the following two conditions:

(a) $\forall v \in [t,T], p(v) \in [0,1]$

(b) $E_{t,z,s_1,s_2}\left[\int_t^T |\pi_1(v)|^4 dv\right] < \infty, E_{t,z,s_1,s_2}\left[\int_t^T |\pi_2(v)|^4 dv\right] < \infty$

where $E_{t,z,s_1,s_2,r}[\cdot] = E[\cdot|Z(t)=z, S_1(t)=s_1, S_2(t)=s_2, r(t)=r]$. In subsequent text, we denote by $U(t,z,s_1,s_2,r)$ the set of all admissible strategies policies with respect to the initial conditions $(t,z,s_1,s_2,r) \in [t,T]$. Based on conditions (a) and (b) and the boundedness of parameters, it follows that:

$$\begin{aligned} E_{t,z,s_1,s_2}[\int_t^T (|(\alpha m\theta - m\eta p(v)(\alpha-\beta)|^2 + |\alpha\pi_1(v)(\mu_1 - r_0))dt|^2 \\ + |\beta\pi_2(v)(\mu_2 - r_0)|^2)dv] < \infty \end{aligned} \quad (11)$$

$$E_{t,z,s_1,s_2}[\int_t^T |\left(\alpha m\theta - m\eta p(v)(\alpha-\beta) + \alpha\pi_1(v)(\mu_1 - r_0)\right)dt + \beta\pi_2(v)(\mu_2 - r_0)|^2 dv] < \infty$$

Therefore, the differential equation representing the wealth process has a unique strong solution for any arbitrary $\rho\in[1,\infty]$ and condition $(t,z,s_1,s_2,r) \in [0,T]\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+$ satisfies the following conditions [18]:

$$E_{t,z,s_1,s_2}\left[\sup_{\mu\in U(t,T)} |Z^{\mu}(t)|^{\rho}\right] < \infty \quad (12)$$

3. Optimal investment strategy

This paper considers the dynamic mean-variance criterion. For any $(t,z,s_1,s_2,r) \in [0,T]\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+$, the optimization objective of the insurance groups is:

$$\sup_{\mu\in U(t,z,s_1,s_2)} \left\{E_{t,z,s_1,s_2}[Z^{\mu}(t)] - \frac{\gamma}{2}\mathrm{var}_{t,z,s_1,s_2}[Z^{\mu}(t)]\right\} \quad (13)$$

where U represents the corresponding set of admissible strategies, and $\gamma > 0$ is the investors' risk aversion coefficient. Since the variance object lacks the property of iterative expectation, the **Equation 11** does not satisfy the Bellman optimality principle, making it time-inconsistent. To find a time-consistent solution for **Equation 11**, we resort to the equilibrium strategy defined in **Definition 3** below.

In the time-consistent strategy (**Definition 2**), we use game theory methods to solve equilibrium strategies for mean-variance problems by studying a more general problem. Let $Q = \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$. $C^{1,2}([0,T] \times Q) = \{\phi(t,z) | \phi(t,z)$ is a function of the first parameter t, differentiable on $[0, T]$, and the remaining parameters are quadratically differentiable on Q, and $D^{1,2}([0,T] \times Q) = \{\phi(t,z) | \phi(t,z) \in C^{1,2}([0,T] \times Q)$ satisfies the polynomial growth (PG) condition.

The general problem for arbitrary functions is defined as follows:

$$\sup_{\mu \in U(t,z,s_1,s_2)} f\{f, z, s_1, s_2, g^\mu(t,z,s_1,s_2), h^\mu(t,z,s_1,s_2)\} \tag{14}$$

which $(t,z) \in [0,T] \times Q, g^\mu(t,z) = E_{t,z}[Z^\mu(T)], h^\mu(t,z) = E_{t,z}\left[(Z^\mu(T))^2\right]$.

Here, $U(t,z,s_1,s_2)$ is a set of admissible strategies for the state (t,z,s_1,s_2), with its precise definition given in **Definition 1**. In particular, according to the following equation:

$$f(t,z,s_1,s_2,g,h) = g - \frac{\gamma}{2(h - g^2)} \tag{15}$$

for the time-consistent solution to dynamic problems, the following definition of equilibrium strategy is established.

In the equilibrium strategy (**Definition 3**), given an admissible strategy $\mu^*(t)=(\pi_1^*(t),\pi_2^*(v),p^*(t))\in U(t,z,s_1,s_2)$, the following strategy can be constructed:

Here, $\widetilde{\pi_1}, \widetilde{\pi_2} \in \mathbb{R}, \tilde{p} \in [0,1] v > 0$, if $\lim_{\tau \to 0} \inf \frac{f\left(t,z,s_1,s_2,g^{\mu^*},h^{\mu^*}\right) - f(t,z,s_1,s_2,g^{\mu_\tau},h^{\mu_\tau})}{\tau} \geq 0$ for any $(\pi_1^*(t), \pi_1^*(t), p^*(t)) \in \mathbb{R} \times \mathbb{R} \times [0,1]$ and $(t,z,s_1,s_2) \in [0,T] \times Q$, then we say μ^* is an equilibrium strategy, and correspondingly, the equilibrium value function is defined as:

$$W(t,z,s_1,s_2;\mu^*) = f\left(t,z,s_1,s_2,g^{\mu^*}(t,z,s_1,s_2),h^{\mu^*}(t,z,s_1,s_2)\right) \tag{16}$$

To solve the equilibrium strategy for the mean-variance problem, we establish a verification theorem, which gives the extended HJB equation for the general problem. For any $\phi(t,z,s_1,s_2) \in C^{1,2}([0,T] \times Q)$, the variational operator is defined as follows:

$$\begin{aligned}\mathcal{A}^\mu \phi_t(t,z,s_1,s_2) = \; & \phi_t + [rz + \alpha\pi_1(\mu_1 - r) + \alpha\pi_2(\mu_2 - r) + m\alpha\theta - m\eta p(\alpha - \beta)]\phi_z \\ & + \mu_1 s_1 \phi_{s_1} + \mu_2 s_2 \phi_{s_2} \\ & + \frac{1}{2}\left\{\alpha^2\pi_1^2\sigma_1^2 s_1^{2\beta_1} + \beta^2\pi_2^2\sigma_2^2 s_2^{2\beta_2} + [\alpha n - np(\alpha - \beta)]^2\right\}\phi_{zz} \\ & + \frac{1}{2}\sigma_1^2 s_1^{2\beta_1+2}\phi_{s_1 s_1} + \frac{1}{2}\sigma_2^2 s_2^{2\beta_2+2}\phi_{s_2 s_2} + \alpha\pi_1\sigma_1^2 S_1^{2\beta_1+1}\phi_{zs_1} \\ & + \beta\pi_2\sigma_2^2 S_2^{2\beta_2+1}\phi_{zs_2} \\ & + \lambda_0 E\left[\phi\left(t, z - y_0(\alpha - \alpha p + \beta p), s_1, s_2 - \phi(t,z,s_1,s_2)\right)\right]\end{aligned} \tag{17}$$

In the verification theorem (**Theorem 1**), if there exist two real-valued functions $F(t,z,s_1,s_2), g(t,z,s_1,s_2) \in D^{1,2}([0,T] \times Q)$ satisfying the following equation:

$$\sup_{\mu \in U(t,z,s_1,s_2)} \{\{\mathcal{A}^\mu F(t,z,s_1,s_2) + \gamma g \mathcal{A}^\mu g(t,z,s_1,s_2)\} - \mathcal{A}^\mu \frac{1}{2}\left(g(t,z,s_1,s_2)\right)^2\} = 0 \tag{18}$$

where $F(t, z, s_1, s_2) = z$,

$$\mathcal{A}^{\mu^*} g(t,z,s_1,s_2) = 0, g(t,z,s_1,s_2) = 0 \tag{19}$$

$$\mu^* = \arg \sup_{\mu \in U(t,z,s_1,s_2)} \{\mathcal{A}^\mu F(t,z,s_1,s_2) + \gamma g \mathcal{A}^\mu g(t,z,s_1,s_2)\} - \mathcal{A}^\mu \frac{\gamma}{2}\left(g(t,z,s_1,s_2)\right)^2 \tag{20}$$

then $W(t, z, s_1, s_2) = F(t, z, s_1, s_2)$ and $E_{t, z, s1, s2}[Z^{\mu^*}(T)] = g(t, z, s_1, s_2)$. Hence, the equilibrium strategy is as follows [18]:

$$\mu^* = (\pi_1^*(t), \pi_2^*(t), p^*) \tag{21}$$

4. Temporal consistency strategy under the mean-variance criterion

In this section, we derive the display solutions for the equilibrium strategy and the equilibrium value function. Suppose that a sum exists that satisfies the given conditions $F(t, z, s_1, s_2)$ and $g(t, z, s_1, s_2)$ in **Theorem 1**. According to the expression $\mathcal{A}^\mu$, the **Equations 18** and **19** can be rewritten as:

$$\begin{aligned}\sup_{\mu\in U(t,z,s_1,s_2)} = \{&F_t + [rz + \alpha\pi_1(\mu_1 - r) + \alpha\pi_2(\mu_2 - r)m\alpha\theta - m\eta p(\alpha-\beta)]F_z + \mu_1 s_1 F_{s_1}\\ &+ \mu_2 s_2 F_{s_2} + \frac{1}{2}\left\{\alpha^2\pi_1^2\sigma_1^2 s_1^{2\beta_1} + \beta^2\pi_2^2\sigma_2^2 s_2^{2\beta_2} + [\alpha n - np(\alpha-\beta)]^2\right\}F_{zz}\\ &+ \frac{1}{2}\sigma_1^2 s_1^{2\beta_1+2}F_{s_1 s_1} + \frac{1}{2}\sigma_2^2 s_2^{2\beta_2+2}F_{s_2 s_2} + \alpha\pi_1\sigma_1^2 S_1^{2\beta_1+1}F_{zs_1} + \beta\pi_2\sigma_2^2 S_2^{2\beta_2+1}F_{zs_2}\\ &+ \lambda_0 E\left[F\left(\mathrm{t}, z - y_0(\alpha - \alpha p + \beta p), s_1, s_2 - F(t,z,s_1,s_2)\right)\right]\\ &- \frac{\gamma}{2}\left\{\alpha^2\pi_1^2\sigma_1^2 s_1^{2\beta_1} + \beta^2\pi_2^2\sigma_2^2 s_2^{2\beta_2} + [\alpha n - np(\alpha-\beta)]^2 G_z^2\right\} - \frac{\gamma}{2}\sigma_1^2 s_1^{2\beta_1+2}G_{s_1}^2\\ &- \frac{\gamma}{2}\sigma_2^2 s_2^{2\beta_2+2}G_{s_2}^2 - \gamma\alpha\pi_1\sigma_1^2 S_1^{2\beta_1+1}G_{zs_1} - \beta\pi_2\sigma_2^2 S_2^{2\beta_2+1}G_{zs_2} = 0\end{aligned} \tag{22}$$

$$\begin{aligned}G_t &+ [rz + \alpha\pi_1(\mu_1 - r) + \beta\pi_2(\mu_2 - r) + m\alpha\theta - m\eta p(\alpha-\beta)]G_z + \mu_1 s_1 G_{s_1} + \mu_2 s_2 G_{s_2}\\ &+ \frac{1}{2}\left\{\alpha^2\pi_1^2\sigma_1^2 s_1^{2\beta_1} + \beta^2\pi_2^2\sigma_2^2 s_2^{2\beta_2} + [\alpha n - np(\alpha-\beta)]^2\right\}G_{zz} + \frac{1}{2}\sigma_1^2 s_1^{2\beta_1+2}G_{s_1 s_1}\\ &+ \frac{1}{2}\sigma_2^2 s_2^{2\beta_2+2}G_{s_2 s_2} + \alpha\pi_1\sigma_1^2 S_1^{2\beta_1+1}G_{zs_1} + \beta\pi_2\sigma_2^2 S_2^{2\beta_2+1}G_{zs_2}\\ &+ \lambda_0 E\left[G\left(\mathrm{t}, z - y_0(\alpha - \alpha p + \beta p), s_1, s_2 - G(t,z,s_1,s_2)\right)\right] = 0\end{aligned} \tag{23}$$

Theorem 2: When $\alpha \neq \beta$, the equilibrium investment strategies for the insurance companies and reinsurance companies are as follows:

$$\pi_1^* = \frac{(\mu_1 - \mathrm{r})\gamma + (\mu_1 - \mathrm{r})^2\left[1 - e^{-2\gamma\beta_1(T-t)}\right]}{\alpha\sigma_1^2 S_1^{2\beta} e^{r(T-t)}} \tag{24}$$

$$\pi_2^* = \frac{(\mu_2 - \mathrm{r})\gamma + (\mu_2 - \mathrm{r})^2\left[1 - e^{-2\gamma\beta_2(T-t)}\right]}{\alpha\sigma_2^2 S_2^{2\beta_2} e^{r(T-t)}} \tag{25}$$

The equilibrium reinsurance strategy for the insurance company is:

$$p^* = \frac{\alpha}{\alpha-\beta} - \frac{m\eta}{(\alpha-\beta)n^2\gamma}e^{-r(T-t)}, 0 < t < T \tag{27}$$

Additionally, the equilibrium value function is: $V(t, z, s_1, s_2) = A_1(t)z + B_1(t)s_1 + C_1(t)s_2 + D_1(t)$, where $A_1(t)$, $B_1(t)$, $C_1(t)$ and $D_1(t)$ are provided in **Appendix 1**.

Theorem 3: When $\alpha=\beta$, **Equations 24** and **25** remain the equilibrium investment strategies for the insurance companies and reinsurance companies. Furthermore, for any measurable function $p^*:[0,\mathrm{T}] \to [0,1]$, it is a solution to the equilibrium reinsurance strategy. In this case, the equilibrium value function is given by:

$$V(t, z, s_1, s_2) = A_1(t)z + B_1(t)s_1 + C_1(t)s_2 + N_1(t) \tag{28}$$

where $N_1(\mathrm{t}) = \frac{\gamma m\alpha(\theta-\eta)}{r}\left(e^{r(T-t)} - 1\right) + \lambda_0\mu_y \frac{m\eta}{n^2\gamma^2}(T-t) + \sigma_1^2\beta_1(2\beta_1+1)B_1(t)\int_t^T B_2(s)ds$.

Notably, **Theorem 2** indicates that equilibrium strategies depend on the decision weights α and β, as they exist in the weighted summation process. The equilibrium reinsurance strategy given by the theorem is as follows:

$$p^* = \frac{\alpha}{\alpha-\beta} - \frac{m\eta}{(\alpha-\beta)n^2\gamma}e^{-r(T-t)} \tag{29}$$

Therefore, there are two interpretations of the relationship between reinsurance strategy and α. On the one hand, α/(α-β) decreases the value of p^* with increasing α; on the other hand, $[-m\eta/(\alpha-\beta)b^2\gamma e^{-r(T-t)}]$ increases the value of p^* with increasing α. Furthermore, **Equations 24** and **25** indicate that α negatively impacts the investment strategy of the insurance company but positively affects the strategy of reinsurance companies.

Based on the results established in **Theorem 2**, the sensitivity of the equilibrium reinsurance ratio $p^*(t)$ to various model parameters can be analyzed as follows:

(1) According to **Equation 29**, we obtain:

$$\frac{\partial p^*}{\partial \alpha} = \frac{1}{(\alpha-\beta)^2}\left[\frac{m\eta}{n^2\gamma}e^{-r(T-t)} - \beta\right], \frac{\partial p^*}{\partial \beta} = \frac{1}{(\alpha-\beta)^2}\left[\alpha - \frac{m\eta}{n^2\gamma}e^{-r(T-t)}\right] \tag{30}$$

When $\beta < e^{-r(T-t)}m\eta/n^2\gamma$, the optimal reinsurance equilibrium strategy increases with α, and when $\alpha > \frac{m\eta e^{-r(T-t)}}{n^2\gamma}$, the optimal reinsurance equilibrium strategy increases with β.

(2) It is easy to deduce:

$$\frac{\partial p^*}{\partial t} = -\frac{m\eta r}{(\alpha-\beta)b^2\gamma}e^{-r(T-t)} \tag{31}$$

This implies that the equilibrium reinsurance strategy when $\alpha > \beta$ decreases with increasing time t, indicating a greater preference for insurance companies in decision-making. Therefore, according to the insurance companies' intention, insurance companies retain more insurance business at later periods, leading to a decrease in the reinsurance ratio as time t changes. The equilibrium reinsurance strategy when $\alpha < \beta$ increases with increasing time t. As time passes, both insurance companies and reinsurance companies expect the potential returns on financial asset investments to increase on average, which will enhance their ability to absorb insurance risks.

(3) The impact of market parameters on reinsurance strategies also depends on the decision weights α and β.

$$\frac{\partial p^*}{\partial m} = -\frac{\eta e^{-r(T-t)}}{(\alpha-\beta)^2 n^2\gamma} \tag{32}$$

This means that when $\alpha < \beta$, the equilibrium reinsurance strategy increases with m, while when $\alpha < \beta$, the equilibrium reinsurance strategy decreases with m. The parameter m reflects the expected claim size, so, holding other model parameters fixed (especially the volatility n in the claim process **Definition 1**), increasing the expected claims reduces the risk per dollar of insurance liability, making the insurance business more attractive. In contrast, if $\alpha < \beta$, more weight is given to reinsurance companies in decision-making, and more policies are transferred to reinsurance companies.

(4) Due to the following equation:

$$\frac{\partial p^*}{\partial \eta} = -\frac{m e^{-r(T-t)}}{(\alpha-\beta)^2 n^2\gamma} \tag{33}$$

the safety load η of reinsurance companies has a positive effect on $p^*(\cdot)$ for $\alpha<\beta$ and a negative effect on $p^*(\cdot)$ for $\alpha>\beta$. This is consistent with our intuition, as a larger η implies more expensive reinsurance. Therefore, when $\alpha<\beta$, decision-makers will consider reinsurance companies more, thereby increasing the reinsurance ratio, allowing reinsurance companies to gain more profit with increasing η.

(5) **Equation 27** also indicates:

$$\frac{\partial p^*}{\partial \gamma}=\frac{m\eta e^{-r(T-t)}}{(\alpha-\beta)^2 n^2 \gamma^2} \tag{34}$$

showing that the equilibrium reinsurance ratio $p^*(\cdot)$ when $\alpha<\beta$ decreases with increasing risk aversion coefficient γ of the decision-maker, while the equilibrium reinsurance ratio for $\alpha>\beta$ increases with increasing γ. For decision weights $\alpha<\beta$, decision-making relies more on reinsurance companies' preference than insurance companies. Therefore, the greater the risk aversion of reinsurance companies, the less reinsurance they undertake, resulting in a decrease in $p^*(\cdot)$ with γ.

(6) **Equation 27** also indicates:

$$\frac{\partial p^*}{\partial r}=-\frac{m\eta(T-t)e^{-r(T-t)}}{(\alpha-\beta)^2 n^2 \gamma} \tag{34}$$

This indicates that $p^*(\cdot)$ relative to the risk-free rate r decreases for $\alpha<\beta$ and increases for $\alpha>\beta$. With increasing r, both insurance companies and reinsurance companies expect to earn more returns in financial markets, driving capital out of the insurance market. When $\alpha<\beta$, the interests of the reinsurance companies dominate the decision on insurance group transaction strategy, leading reinsurance companies to reduce their investment in the insurance market and transfer more investment to financial markets.

5. Numerical analysis

This section will provide some numerical examples to illustrate the impact of parameters on the equilibrium investment-reinsurance strategy derived from the theorem. Unless otherwise stated, all parameters in this section are set as follows: $m = 0.5$, $n = 0.6$, $\eta = 0.2$, $r = 0.05$, $\mu_1 = 0.12$, $\sigma_1 = 0.2$, $\beta_1 = 0.9$, $s_1 = 0.5$, $\mu_2 = 0.15$, $\sigma_2 = 0.3$, $\beta_2 = 1.1$, $s_2 = 0.6$, $\alpha = 0.3$, $\beta = 0.7$, $\gamma = 0.5$, and $T = 10$.

Figure 1 illustrates that when $\alpha<\beta$, the optimal reinsurance ratio $p^*(t)$ increases with time t, while when $\alpha>\beta$, $p^*(t)$ decreases with time t. This phenomenon can be explained as follows: as the potential returns on financial assets increase, the willingness of insurance companies and reinsurance companies to take risks gradually increases over time. Therefore, both parties tend to absorb less insurance risk in the first stage and more insurance business in the second stage. Thus, when $\alpha<\beta$, reinsurance companies have more say, and more insurance business is transferred to reinsurance companies in the later stage, leading to an increase in the reinsurance ratio over time. Conversely, when $\alpha>\beta$, insurance groups prioritize reinsurance profits, and insurance risk is transferred to reinsurance companies earlier.

Figures 2–5 represent the impact of market parameters on the equilibrium reinsurance strategy, confirming the comments made in the notes. These impacts also depend on the values of the decision weight parameters α and β. When $\alpha<\beta$, the equilibrium reinsurance ratio $p^*(t)$ increases with increasing γ, m, and r parameters, and decreases with increasing t, η, and n parameters; conversely, when $\alpha>\beta$, $p^*(t)$ decreases with increasing γ, m, and r parameters, and increases with increasing t, η, and n parameters.

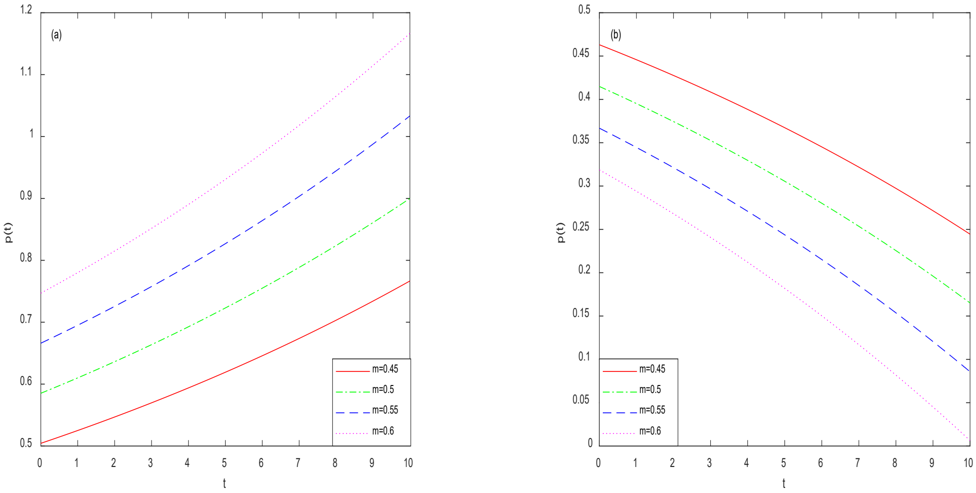

Figure 1. The α and β effect of decision weights on the optimal reinsurance ratio

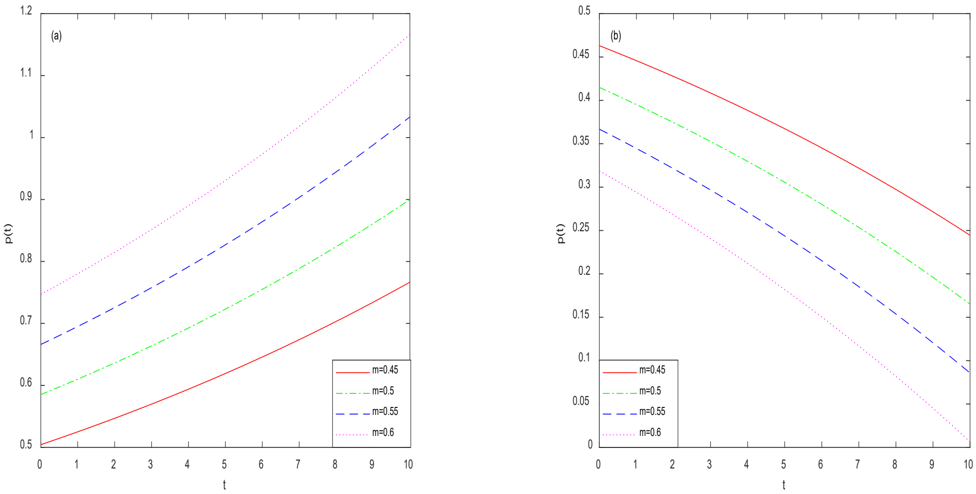

Figure 2. The *m* effect on the optimal reinsurance ratio

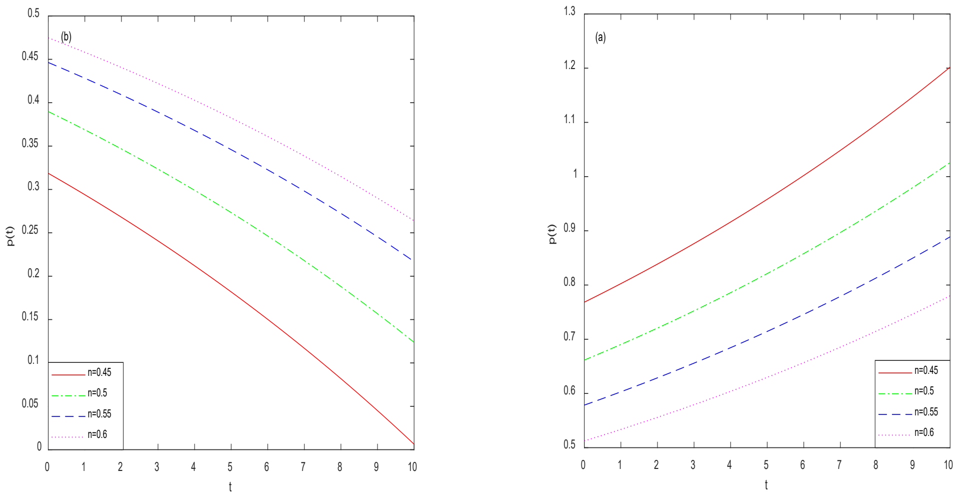

Figure 3. The *n* effect on the optimal reinsurance ratio

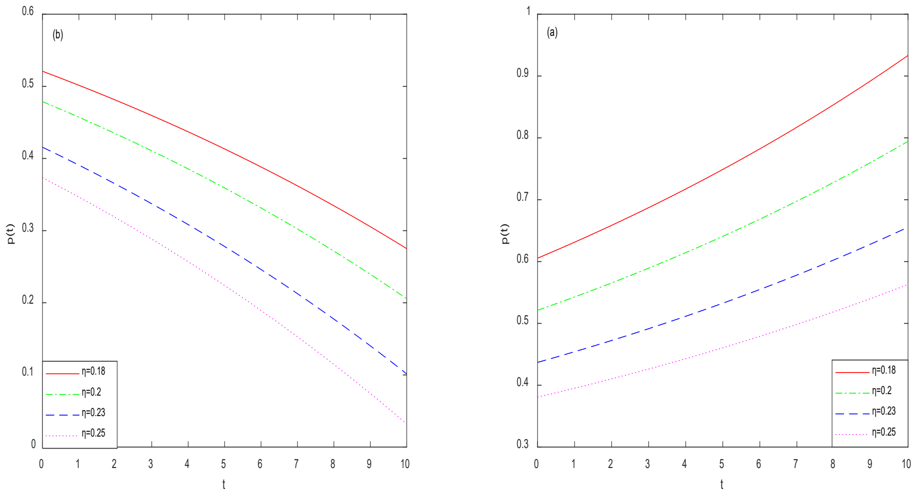

Figure 4. The η effects on the optimal reinsurance ratio

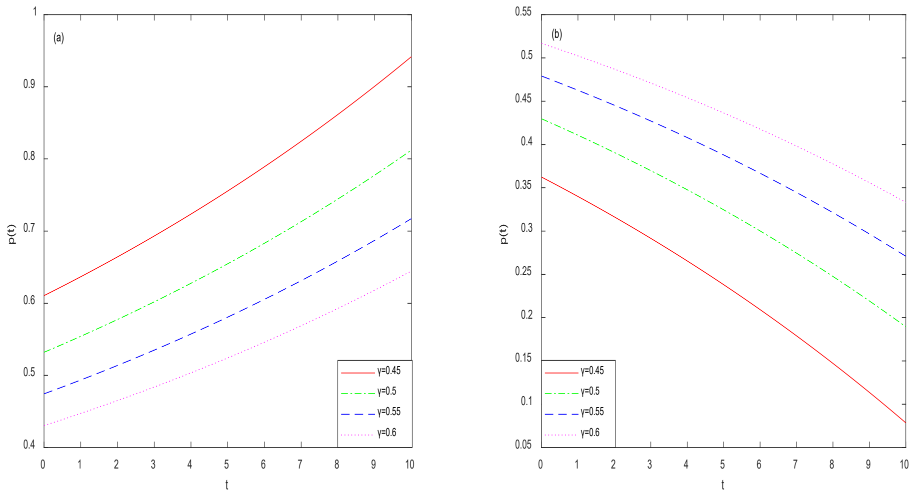

Figure 5. The γ effects on the optimal reinsurance ratio

Figures 6–9 depict the sensitivity analysis of the equilibrium investment strategy (π_1^*, π_2^*) to various parameters. To simplify representation without losing generality, only the equilibrium investment strategy at time $t = 0$ is considered.

Figure 6 (left) illustrates that the risk aversion coefficient γ has a negative impact on both π_1^* and π_2^*. A higher γ indicates greater aversion to risk by insurance companies and reinsurance companies. Thus, as γ increases, these companies choose to reduce their holdings of risky assets to control risk.

Figure 6 (right) describes the adverse effect of the risk-free assets return rate r on the equilibrium investment strategy (π_1^*, π_2^*). An increase in the risk-free rate expands investment opportunities for insurance companies and reinsurance companies in the financial market. Consequently, they are more inclined to invest in risk-free assets and reduce investments in reinsurance assets.

Figures 7 and **9** describe the parameters μ_1, μ_2 and β_1, β_2 on the optimal investment strategy π_1^* and π_2^*. This indicates that π_1^* and π_2^* are increasing functions of μ_1, μ_2 and β_1, β_2, where μ_1, μ_2 represent the expected returns of risk assets. A higher value of μ_1, μ_2 signifies higher expected returns of risky assets. Therefore, insurance companies and reinsurance companies increase their investments in risky assets to obtain greater returns, leading to an increase in the optimal investment strategy π_1^* and π_2^*. The negative impact of the volatility of risky assets on the optimal investment strategy π_1^* and π_2^* in **Figure 8** can also be explained by the same theoretical principles.

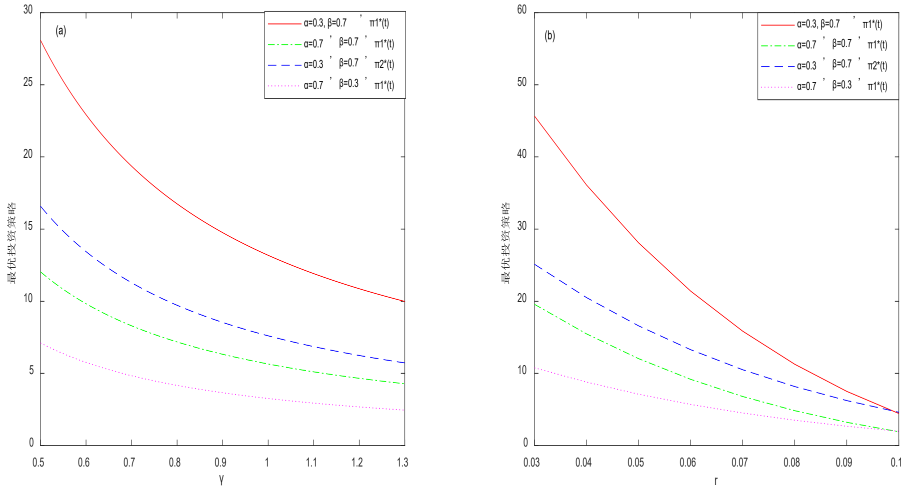

Figure 6. (left) The γ impact on (π_1^*,π_2^*); **(right)** The r impact on (π_1^*,π_2^*)

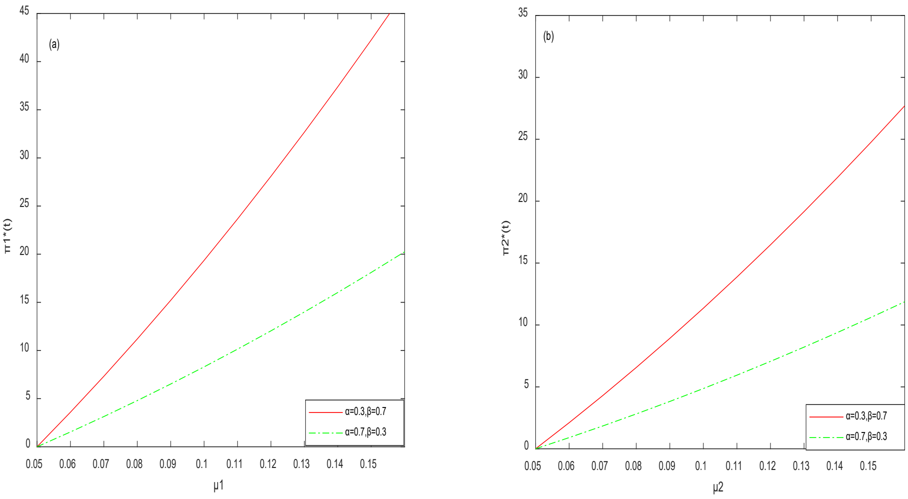

Figure 7. (left) The μ_1 impact on (π_1^*,π_2^*); **(right)** The μ_2 impact on (π_1^*,π_2^*)

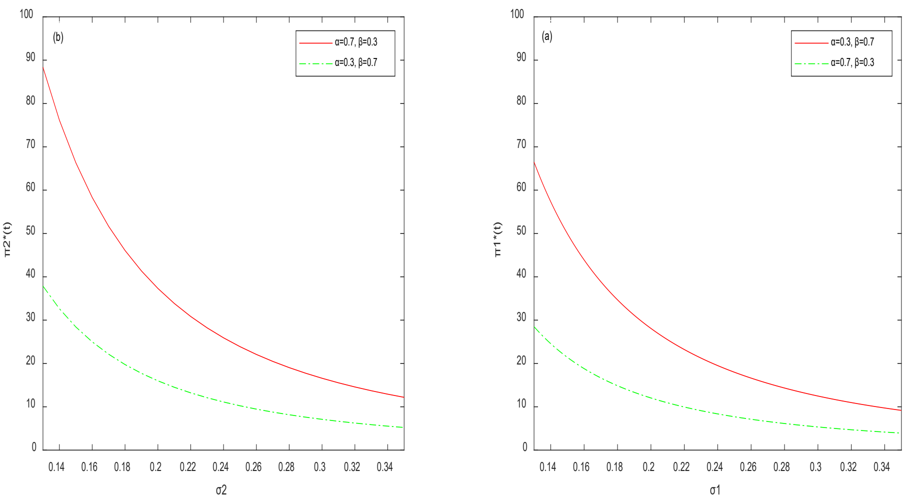

Figure 8. (left) The σ_1 impact on (π_1^*,π_2^*); **(right)** The σ_2 impact on (π_1^*,π_2^*)

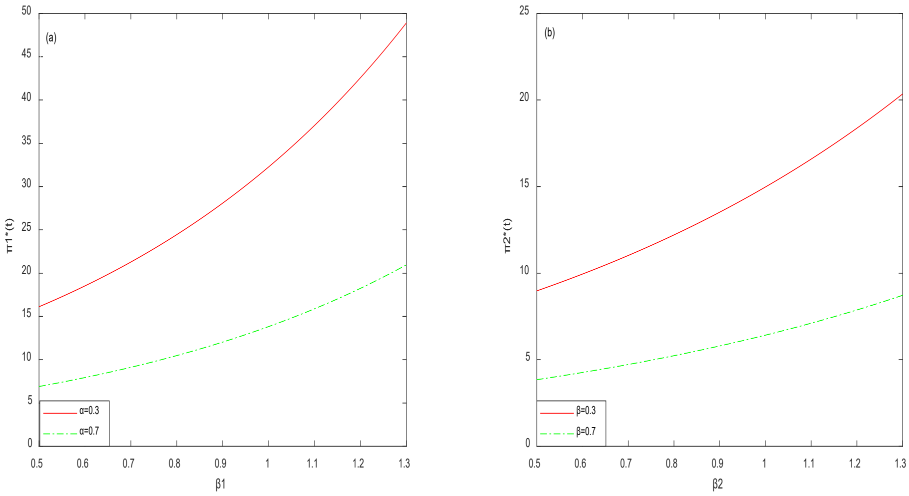

Figure 9. (left) The β_1 impact on (π_1^*,π_2^*); **(right)** The β_2 impact on (π_1^*,π_2^*)

6. Conclusion

From the perspective of insurance companies and reinsurance companies, using game theory and citing auxiliary functions and verification theorems, we establish a value function that satisfies the extended HJB equation. We guess the form of the solution, obtain the optimal investment and reinsurance strategy under the mean-variance optimization target, and conduct a sensitivity analysis of each parameter. The results show that: (1) when the decision weight of insurance companies is greater, and as the risk aversion factor increases, the expected increase in claim size makes the insurance business even more attractive. Insurance companies will thus reduce the number of reinsurances; (2) the influence of the variance coefficient of elasticity on the optimal investment strategy is positive. The greater this parameter, the higher the probability of risky asset prices, prompting insurers and reinsurers to increase their positions in risky assets; (3) when the insurance company has decision-making power, the safety load of the reinsurance company has a negative effect on the insurance company. Greater safety loads represent higher reinsurance costs, leading insurance companies to reduce the reinsurance ratio.

This paper represents only a preliminary study on the joint benefits of insurance groups, and there are still many issues to be further explored. For example, fuzzy aversion can be introduced based on studying the interests of insurance groups.

Disclosure statement

The authors declare no conflict of interest.

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Appendix A

Proof: Due to the linear structure, according to the boundary conditions, we attempt to guess the solution as follows:

$$V(t,z,s_1,s_2) = A_1(t)z + B_1(t)s_1^{-2\beta} + C_1(t)s_2^{-2\beta_2} + \frac{D_1(t)}{\gamma}$$
$$g(t,z,s_1,s_2) = A_2(t)z + B_2(t)s_1^{-2\beta} + C_2(t)s_2^{-2\beta_2} + \frac{D_2(t)}{\gamma}$$

Boundary condition:

$$A_1(T) = A_2(T) = 1, B_1(T) = C_1(T) = D_1(T) = B_2(T) = C_2(T) = D_2(T) = 0$$
$$F_t = A_1^{'}(t)z + \overset{B^{'}(t)}{-}(C_1 s_1^{-2\beta_1} + \frac{C_1^{'}(t)}{\gamma}s_2^{-2\beta_2} + \frac{D_1^{'}(t)}{\gamma}, F_z = A_1(t),$$
$$F_{s_2} = \frac{-2\beta_1 C_1(t)}{\gamma}s_2^{-2\beta_2-1}, F_{s_1 s_1} = \frac{2\beta_1(2\beta_1+1)B_1(t)}{\gamma}s_1^{-2\beta_1-1},$$
$$F_{s_2 s_2} = \frac{2\beta_2(2\beta_2+1)C_1(t)}{\gamma}s_1^{-2\beta_2-1}, F_{zz} = F_{s_1} = F_{s_2} = F_{s,s_1} = F_{s_2 s_2} = 0;$$
$$G_t = A_2^{'}(t)z + \frac{B_2^{'}(t)}{\gamma}s_1^{-2\beta_1} + \frac{C_2^{'}(t)}{\gamma}s_2^{-2\beta_2} + \frac{D_2^{'}(t)}{\gamma}, G_z = A_2(t),$$

$$G_{s_1} = \frac{-2\beta_1 B_2(t)}{\gamma}s_1^{-2\beta_1-1}, G_{s_2} = \frac{-2\beta_2 C_2(t)}{\gamma}s_2^{-2\beta_2-1}$$
$$G_{s_1 s_1} = \frac{2\beta_1(2\beta_1+1)B_2(t)}{\gamma}s_1^{-2\beta_1-1}, G_{s_2 s_2} = \frac{2\beta_2(2\beta_1+1)C_2(t)}{\gamma}s_2^{-2\beta_2-1}$$
$$G_{:=} = G_{\approx 1} = G_{z=2} = G_{s_1 s_1} = G_{s_2 s_2} = 0.$$

Substituting the above derivatives to **Equations 22** and **23**:

$$\begin{aligned}\sup_{\mu\in U(t,z,s_1,s_2)} = \ & \{[rz + \alpha\pi_1(\mu_1 - r) + \alpha\pi_2(\mu_2 - r) + m\alpha\theta - m\eta p(\alpha-\beta)]A_1(t) \\ & + A_1^{'}(t)z + \frac{B_1^{'}(t)}{\gamma}s_1^{-2\beta_1} + \frac{C_1^{'}(t)}{\gamma}s_2^{-2\beta_2} + \frac{D_1^{'}(t)}{\gamma} \\ & + \mu_1 s_1\frac{-2\beta_1 B_1(t)}{\gamma}s_1^{-2\beta_1-1} + \mu_2 s_2\frac{-2\beta_1 C_1(t)}{\gamma}s_2^{-2\beta_2-1} \\ & + \frac{1}{2}\sigma_1^2 s_1^{2\beta_1+2}\frac{2\beta_1(2\beta_1+1)B_1(t)}{\gamma}s_1^{-2\beta_1-1} \\ & + \frac{1}{2}\sigma_2^2 s_2^{2\beta_2+2}\frac{2\beta_2(2\beta_2+1)C_1(t)}{\gamma}s_1^{-2\beta_2-1} \\ & +\gamma g(t,z,s_1,s_2)\lambda_0 E[g(\mathrm{t}, z - y_0(\alpha - \alpha p + \beta p), s_1, s_2) - g(t,z,s_1,s_2)] \\ & - \lambda_0 E[F(\mathrm{t}, z - y_0(\alpha - \alpha p + \beta p), s_1, s_2) - F(t,z,s_1,s_2)] - \\ & \frac{\gamma}{2}\{\alpha^2\pi_1^2\sigma_1^2 s_1^{2\beta_1} + \beta^2\pi_2^2\sigma_2^2 s_2^{2\beta_2} + [\alpha n - np(\alpha-\beta)]^2 A_z^2\} \\ & + \frac{2\sigma_1^2\beta_1^2 B_2^2(t)}{\gamma}s_1^{-2\beta_1} + \frac{2\sigma_2^2\beta_2^2 C_2^2(t)}{\gamma}s_2^{-2\beta_2}\Bigg\} = 0\end{aligned} \quad \text{(A1)}$$

$$
\begin{aligned}
&[rz + \alpha\pi_1(\mu_1 - r) + \alpha\pi_2(\mu_2 - r) + m\alpha\theta - m\eta p(\alpha - \beta)]A_2(t) \\
&+ A_2^{'}(t)z + \frac{B_2^{'}(t)}{\gamma}s_1^{-2\beta_1} + \frac{C_2^{'}(t)}{\gamma}s_2^{-2\beta_2} + \frac{D_2^{'}(t)}{\gamma} \\
&+ \mu_1 s_1 \frac{-2\beta_1 B_2(t)}{\gamma}s_1^{-2\beta_1 - 1} + \mu_2 s_2 \frac{-2\beta_1 C_2(t)}{\gamma}s_2^{-2\beta_2 - 1} \\
&+ \frac{1}{2}\sigma_1^2 s_1^{2\beta_1 + 2}\frac{2\beta_1(2\beta_1 + 1)B_2(t)}{\gamma}s_1^{-2\beta_1 - 1} + \frac{1}{2}\sigma_2^2 s_2^{2\beta_2 + 2}\frac{2\beta_2(2\beta_2 + 1)C_2(t)}{\gamma}s_1^{-2\beta_2 - 1} \\
&- \lambda_0 E[G(\mathrm{t}, z - y_0(\alpha - \alpha p + \beta p), s_1, s_2) - G(t, z, s_1, s_2)] \\
&\sup_{\mu \in U(t,z,s_1,s_2)} = 0
\end{aligned}
\tag{A2}
$$

Differentiating **Equation A1** with respect to π_1, π_2, and p, we obtain the following first-order optimality conditions:

$$
\begin{cases}
\pi_1^* = \dfrac{(\mu_1 - r)A_1(t) + 2\beta_1 B_2(t)\sigma_1^2 A_2(t)}{\alpha\sigma_1^2 s_1^{2\beta_1}\gamma A_2^2(t)} \\
\pi_2^* = \dfrac{(\mu_2 - r)A_1(t) + 2\beta_2 C_2(t)\sigma_2^2 A_2(t)}{\beta\sigma_2^2 s_2^{2\beta_2}\gamma A_2^2(t)}
\end{cases}
\tag{A3}
$$

$$
p^* = \frac{\alpha}{\alpha - \beta} - \frac{m\eta}{(\alpha - \beta)n^2\gamma}e^{-r(T-t)} \tag{A4}
$$

Substituting **Equations A3** and **A4** into **Equations A1** and **A2** and rearranging, we obtain:

$$
\begin{aligned}
\sup_{\mu \in U(t,z,s_1,s_2)} &= \Big\{\big(rA_1(t) + A_1^{'}(t)\big)z + \\
&\Bigg[\frac{(\mu_1 - r)^2 A_1(t)A_1(t) + 2(\mu_1 - r)\beta_1 B_2(t)\sigma_1^2 A_2(t)A_1(t)}{\sigma_1^2 A_2^2(t)} + B_1^{'}(t) \\
&-2\mu_1\beta_1 B_1(t) - 2\sigma_1^2\beta_1^2 B_2{}^2(t) \\
&+ \frac{\left[(\mu_1 - r)A_1(t) + 2\beta_1 B_2(t)\sigma_1^2 A_2(t)A_1(t)\right]^2}{2\sigma_1^2 A_2^2(t)}\Bigg]\frac{s_1^{-2\beta}}{\gamma} \\
&+ \Bigg[\frac{(\mu_2 - r)^2 A_1(t)A_1(t) + 2(\mu_2 - r)\beta_2 C_2(t)\sigma_2^2 A_2(t)A_1(t)}{\sigma_2^2 A_2^2(t)} + C_1^{'}(t) \\
&-2\mu_2\beta_2 C_1(t) - 2\sigma_2^2\beta_2^2 C_2^2(t) \\
&+ \frac{\left[(\mu_2 - r)A_1(t) + 2\beta_2 C_2(t)\sigma_2^2 A_2(t)A_1(t)\right]^2}{2\sigma_2^2 A_2^2(t)}\Bigg]\frac{s_2{}^{-2\beta_2}}{\gamma} + \frac{D_1^{'}(t)}{\gamma} \\
&+ \sigma_1^2\frac{2\beta_1(2\beta_1 + 1)B_1(t)}{\gamma} + \sigma_2^2\frac{2\beta_2(2\beta_2 + 1)C_1(t)}{\gamma} + m\alpha A_1(t)(\theta - \eta) \\
&+ \frac{m^2\eta^2 A_1^2(t)}{n^2\gamma A_2^2(t)} - \lambda_0\mu_y\frac{m\eta A_1^2(t)}{n^2\gamma^2 A_2^2(t)} - \frac{\lambda_0\mu_y m^2\eta^2 A_1^2(t)}{2n^2\gamma^2 A_2^2(t)}\Big\} = 0
\end{aligned}
$$

$$\left(rA_2(t)+A_2'(t)\right)z+\left[\frac{(\mu_1-r)^2A_1(t)+2(\mu_1-r)\beta_1B_2(t)\sigma_1^2A_2(t)}{\sigma_1^2A_2(t)}+B_2'(t)\right.$$
$$\left.-2\mu_1\beta_1C_1(t)\right]\frac{s_1^{-2\beta_1}}{\gamma}+\left[\frac{(\mu_2-r)^2A_1(t)+2(\mu_2-r)\beta_2C_2(t)\sigma_2^2A_2(t)}{\sigma_2^2A_2^2(t)}+C_2'(t)-2\mu_2\beta_2C_1(t)\right]\frac{s_2^{-2\beta_2}}{\gamma}$$
$$+\frac{D_2'(t)}{\gamma}+m\alpha A_2(t)(\theta-\eta)+\sigma_1^2\frac{2\beta_1(2\beta_1+1)B_2(t)}{\gamma}+\sigma_2^2\frac{2\beta_2(2\beta_2+1)C_2(t)}{\gamma}$$
$$-\frac{m^2\eta^2A_1^2(t)}{n^2\gamma A_2{}^2(t)}-\lambda_0\mu_y\frac{m\eta A_1^2(t)}{n^2\gamma^2A_2^2(t)}=0$$

$$\sup_{\mu\in U(t,z,s_1,s_2)}=0$$

To eliminate the dependence on z,S_1,S_2, the equation can be decomposed as:

$$rA_1(t)+A_1'(t)=0$$
$$B_1'(t)-2\mu_1\beta_1B_1(t)-2\sigma_1^2\beta_1^2B_2{}^2(t)+\frac{\left[(\mu_1-r)A_1(t)+2\beta_1B_2(t)\sigma_1^2A_2(t)A_1(t)\right]^2}{2\sigma_1^2A_2^2(t)}=0$$
$$C_1'(t)-2\mu_2\beta_2C_1(t)-2\sigma_2^2\beta_2^2C_2{}^2(t)+\frac{\left[(\mu_2-r)A_1(t)+2\beta_2C_2(t)\sigma_2^2A_2(t)A_1(t)\right]^2}{2\sigma_2^2A_2^2(t)}=0$$
$$\frac{D_1'(t)}{\gamma}+\sigma_1^2\frac{2\beta_1(2\beta_1+1)B_1(t)}{\gamma}+\sigma_2^2\frac{2\beta_2(2\beta_2+1)C_1(t)}{\gamma}+m\alpha A_1(t)(\theta-\eta)+\frac{m^2\eta^2A_1{}^2(t)}{n^2\gamma A_2{}^2(t)}$$
$$-\lambda_0\mu_y\frac{m\eta A_1^2(t)}{n^2\gamma^2A_2^2(t)}-\frac{\lambda_0\mu_y m^2\eta^2A_1^2(t)}{2n^2\gamma^2A_2^2(t)}=0$$
$$rA_2(t)+A_2'(t)=0$$
$$\frac{(\mu_1-r)^2A_1(t)+2(\mu_1-r)\beta_1B_2(t)\sigma_1^2A_2(t)}{\sigma_1^2A_2(t)}+B_2'(t)-2\mu_1\beta_1C_1(t)=0$$
$$\frac{(\mu_2-r)^2A_1(t)+2(\mu_2-r)\beta_2C_2(t)\sigma_2^2A_2(t)}{\sigma_2^2A_2^2(t)}+C_2'(t)-2\mu_2\beta_2C_1(t)=0$$
$$\frac{D_2'(t)}{\gamma}+m\alpha A_2(t)(\theta-\eta)+\sigma_1^2\frac{2\beta_1(2\beta_1+1)B_2(t)}{\gamma}+\sigma_2^2\frac{2\beta_2(2\beta_2+1)C_2(t)}{\gamma}-\frac{m^2\eta^2A_1^2(t)}{n^2\gamma A_2^2(t)}$$
$$-\lambda_0\mu_y\frac{m\eta A_1^2(t)}{n^2\gamma^2A_2^2(t)}=0$$

Considering the boundary conditions, we obtain:

$$A_1(t)=e^{r(T-t)}$$
$$B_1(t)=\frac{(\mu_1-r)^2}{2\beta_1\sigma_1^2\gamma}\left[e^{-2\mu_1\beta_1(T-t)}-e^{-2\gamma\beta_1(T-t)}\right]+\frac{(\mu_1-r)^2(2\mu_1-r)}{4\mu_1\beta_1\sigma_1^2\gamma}\left[1-e^{-2\mu_1\beta_1(T-t)}\right]$$
$$C_1(t)=\frac{(\mu_2-r)^2}{2\beta_2\sigma_2^2\gamma}\left[e^{-2\mu_2\beta_2(T-t)}-e^{-2\gamma\beta_2(T-t)}\right]+\frac{(\mu_2-r)^2(2\mu_2-r)}{4\mu_2\beta_2\sigma_2^2\gamma}\left[1-e^{-2\mu_2\beta_2(T-t)}\right]$$
$$D_1(t)=\frac{\gamma m\alpha(\theta-\eta)}{r}\left(e^{r(T-t)}-1\right)+\frac{\lambda_0\sigma_y^2m^2\eta^2}{2n^2\gamma^2}(T-t)+\sigma_1^2\beta_1(2\beta_1+1)B_1(t)\int_t^TB_1(s)ds$$
$$+\sigma_2^2\beta_2(2\beta_2+1)B_2(t)\int_t^TC_1(s)ds$$

$$A_2(t)=e^{r(T-t)}$$
$$B_2(t)=\frac{(\mu_1-r)^2}{2\beta_1\sigma_1^2\gamma}\left[1-e^{-2\mu_1\beta(T-t)}\right]$$
$$C_2(t)=\frac{(\mu_2-r)^2}{2\beta_2\sigma_2^2\gamma}\left[1-e^{-2\mu_2\beta_2(T-t)}\right]$$
$$D_2(t)=\frac{\gamma m\alpha(\theta-\eta)}{r}\left(e^{r(T-t)}-1\right)+\lambda_0\mu_y\frac{m\eta}{n^2\gamma^2}(T-t)+\sigma_1^2\beta_1(2\beta_1+1)B_1(t)\int_t^TB_2(s)ds$$
$$+\sigma_2^2\beta_2(2\beta_2+1)B_2(t)\int_t^TC_2(s)ds$$

When substituting the above formulas into **Equations A1** and **A2**, we get:

$$\begin{cases} \pi_1^* = \dfrac{(\mu_1 - \mathrm{r})\gamma + (\mu_1 - \mathrm{r})^2\left[1 - e^{-2\gamma\beta_1(T-t)}\right]}{\alpha\sigma_1^2 s_1^{2\beta_1} e^{r(T-t)}} \\ \pi_2^* = \dfrac{(\mu_2 - \mathrm{r})\gamma + (\mu_2 - \mathrm{r})^2\left[1 - e^{-2\gamma\beta_2(T-t)}\right]}{\alpha\sigma_2^2 s_2^{2\beta_2} e^{r(T-t)}} \end{cases}$$

$$p^* = \frac{\alpha}{\alpha - \beta} - \frac{m\eta}{(\alpha - \beta)n^2\gamma} e^{-r(T-t)}$$