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Review Article

Modeling of Electrodynamic Systems by the Method of Binary Separation of Additive Parameter in Golden Proportion

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Abstract: In this paper, based on the physical analogy between processes in elastic-dynamic and electrical systems, a mathematical model of binary structuring of the additive parameter is developed on the basis of the golden ratio of the ratio of the divided parts. It is shown that the direct consequence of the use of binary software structuring is the generation of the Fibonacci sequence. For electrodynamic systems of the Newtonian type, a method for calculating the limits values of the mean and the dispersion of the probabilistic distribution of the Fibonacci additive parameters that are binary structured according to the golden ratio.

Keywords: *Elastic-dynamic and electrical oscillator, Golden ratio, Fibonacci sequence, Variance, Random variables*

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1 Introduction

Different mathematical approaches are used for simulation of electrical engineering systems, including the division into two unequal parts (binary separation) of additive parameter in golden proportion (GP)[1]

$$\varphi_\Lambda = \frac{\lambda}{\Lambda} = \frac{\Lambda-\lambda}{\lambda} \Rightarrow \varphi^2 + \varphi - 1 = 0 \Rightarrow \varphi_\pm = \frac{-1 \pm \sqrt{5}}{2} \Rightarrow \begin{cases} \varphi_{\Lambda+} = \varphi_\Lambda = \dfrac{-1+\sqrt{5}}{2} \cong +0.618, \\ |\varphi_{\Lambda-}| = \Phi_\Lambda = \dfrac{-1+\sqrt{5}}{2} \cong -1.618. \end{cases} \quad (1.1)$$

This method of structuring the system was used in solving many electrical problems[2-13], and most of these studies used regularities of irrational roots $\varphi_{\Lambda\pm}$ on the basis of the theorem of Vieet as the roots of the characteristic equation

$$\lambda^2 + p\lambda - q = 0, \quad (1.2)$$

homogeneous recurrent second order sequence

$$F_{n+2} = F_{n+1} + F_n,\ n \geq 2. \quad (1.3)$$

The sequence $\{F_n\}$ of numbers (1.2) is a sequence of numbers with the so-called partial $p=1, q=-1$ distribution of Fibonacci, whose members are generated by the power transformation of the quantitative characteristics of GP φ_Λ:

$$\varphi_\Lambda{}^n = F_n\varphi_\Lambda + F_{n-1},\ n \in \mathrm{N}, \quad (1.4)$$

which in a more general form is known as the Horadam distribution[14,15]

$$G_{n+2} = pG_{n+1} + qG_n,\ n \geq 2. \quad (1.5)$$

Since their members are determined by the values of the two previous ones, the first two must be specified and have fixed values. In the case of GP, from (1.3) the initial conditions are written as:

$$\begin{cases} \varphi_\Lambda{}^0 = 0(F_0)\cdot\varphi_\Lambda + 1, \\ \varphi_\Lambda{}^1 = 1(F_1)\cdot\varphi_\Lambda + 0, \\ \varphi_\Lambda{}^2 = -1(F_2)\cdot\varphi_\Lambda + 1, \end{cases} \Rightarrow F_0 = 0, \quad F_1 = 1. \quad (1.6)$$

In this paper, using the analogy between the laws of Hooke and Coulomb, the regularities of binary structuring of electrodynamic systems by the method of GP for a set of values p, q that form a parabola in the plane of Cartesian coordinates $p0q$ are studied.

$$q = p^2 \quad (1.7)$$

with branches in the second $p \rangle 0, q \langle 0$ and third $p \langle 0, q$ quadrants. The distribution of (1.5) solves the problem of computing the basic characteristics of the mean and variance.

2 Theoretical models, results obtained and their discussion

2.1 Mathematical model of binary structuring of electrodynamic systems in GP

According to (1.1), the binary separation of a system parameter Λ in a GP is described by a square transform $\lambda^2 + \Lambda\lambda - \Lambda^2 = \lambda^2 + p\lambda - q = \lambda^2 + p\lambda - p^2 = 0$ and forms a set of points of values p, q in the form of a parabola (1.7) in the second quadrant $p \rangle 0, q \langle 0$. His regularities are expressed by power transformations

$$\begin{cases} \lambda^0 = 1 = 0 \cdot \lambda + 1, \\ \lambda^1 = \lambda = 1 \cdot \lambda + 0, \\ \lambda^2 = -\Lambda\lambda + \Lambda^2, \\ \lambda^3 = -\Lambda\lambda^2 + \Lambda^2\lambda = -\Lambda\left(-\Lambda\lambda + \Lambda^2\right) + \Lambda^2\lambda = 2(-\Lambda)^2\lambda - \Lambda^3, \\ \lambda^4 = 2\Lambda^2\lambda^2 - \Lambda^3\lambda = 2\Lambda^2\left(-\Lambda\lambda + \Lambda^2\right) - \Lambda^3\lambda = 3(-\Lambda)^3\lambda + 2\Lambda^4, \\ \lambda^5 = -3\Lambda^3\lambda^2 + 2\Lambda^4\lambda = -3\Lambda^3\left(-\Lambda\lambda + \Lambda^2\right) + 2\Lambda^4\lambda = 5(-\Lambda)^4\lambda - 3\Lambda^5, \\ \lambda^6 = 5\Lambda^4\lambda^2 - 3\Lambda^5\lambda = 5\Lambda^4\left(-\Lambda\lambda + \Lambda^2\right) - 3\Lambda^5\lambda = 8(-\Lambda)^5\lambda + 5\Lambda^6, \\ \lambda^7 = -8\Lambda^5\lambda^2 + 5\Lambda^6\lambda = -8\Lambda^5\left(-\Lambda\lambda + \Lambda^2\right) + 5\Lambda^6\lambda = 13(-\Lambda)^6\lambda - 8\Lambda^7, \\ \ldots \\ \lambda^{n+1} = \alpha_n\lambda + F_n\Lambda^{n+1} = F_{n+1}(-\Lambda)^n\lambda + F_n\Lambda^{n+1}. \end{cases} \quad (2.1)$$

Consequently, in the process (2.1) a sequence of numbers is generated

$$\{\alpha_n\}: 0, 1(-\Lambda)^0, 1(-\Lambda)^1, 2(-\Lambda)^2, 3(-\Lambda)^3, 5(-\Lambda)^4, 8(-\Lambda)^5, 13(-\Lambda)^6, 21(-\Lambda)^7, 34(-\Lambda)^8, 55(-\Lambda)^9, \ldots, \quad (2.2)$$

what for $\Lambda = 1$ of Fibonacci sequence (1.3)

$$\{F_n\} = \{\alpha_n\}\big|_{\Lambda=1}: \quad 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots \quad (2.3)$$

with initial conditions $\alpha_0 = 0, \ \alpha_1 = 1$. Then the limit $n \to \infty$ of relations $\frac{\alpha_n}{\alpha_{n+1}}$ in (2.2) of neighboring members is equal to:

$$\lim_{n\to\infty} \frac{F_n\Lambda^{n+1}}{F_{n+1}\Lambda^n} = \Lambda \lim_{n\to\infty} \frac{F_n}{F_{n-1}} \cong \Lambda \cdot \lambda_+, \quad (2.4)$$

and relaxes to an equilibrium value $\Lambda \cdot \lambda_+$ (for the Fibonacci distribution to $\lambda_\pm$[1]). Similar patterns show flat oscillations of an oscillator in an environment in which the motor resistance is proportional to its velocity:

$$F_r = r\frac{d\xi}{dt} \Rightarrow \frac{r}{m} = 2\gamma. \quad (2.5)$$

Such mechanical systems in the "power-speed" diagram, but electric in the "voltage-current" diagram (if the voltage drop in the section with the load-bearing resistance is proportional to the current strength) have direct characteristics. The degree of dissipation in physical systems, oscillations in which occur at a frequency, characterizes a fading exponential $\exp(-\gamma \cdot t)$ [16,17]. Similar regularities are valid for the ratio of recursive numbers $\frac{F_n}{F_{n-1}}$[18]. Therefore, for simulating physical systems with parameter fluctuations, we can use the method of probabilistic modeling of the scattering characteristics of random changes in the values of the numbers of sequences (1.5).

Since in the formulation of the model GP (1.1) no restriction was imposed on the value of the separation parameter Λ, then the corresponding analysis is relevant for a set of values p, q, on a parabola (1.7), on which the roots are equal:

$$\lambda(\Lambda, \Lambda^2)_\pm = \frac{\Lambda}{2}\left(-1 \pm \sqrt{5}\right) = \Lambda \cdot \lambda(1,1)_\pm \quad (2.6)$$

or

$$\varphi(p,q)_\pm = \varphi(p,p^2)_\pm = \frac{p}{2}\left(-1 \pm \sqrt{5}\right) = p \cdot \varphi(1,1)_\pm. \quad (2.7)$$

Consider the structuring of systems based on the binary separation of the additive parameter for the GP. Oscillatory processes arise in systems with reactive elements from elastic and electric circuits with lumped parameters[16,17]. In the case of parallel combining them among themselves, in elastic systems, an additive force, and in electric - a charge. For consecutive - such parameters are deformation and electrical voltage.

After the deformation of the magnitude dx of the elastic forces to restore the original state perform work $dA = F \cdot dx = \mu x \cdot dx$. In the electric circuit, the Coulomb force field with energy costs plays a similar role $dA = (\varphi_1 - \varphi_2) \cdot dQ$, where $(\varphi_1 - \varphi_2) = \frac{Q}{C}$. Therefore, for a consecutive (2.8 (a, c)) and parallel (2.8 (b, d)) combinations of elements, the mathematical model of the binary separation for the GP will have the form:

$$\begin{cases} L = x + (L-x): \quad \mu_L = \mu_x + \mu_{L-x}, \mu_x^2 = \mu_L^2 - \mu_L\mu_{L-x}, \quad (a) \\ F = F_x + F_{L-x}: \quad \frac{1}{\mu_L} = \frac{1}{\mu_x} + \frac{1}{\mu_{L-x}}, \frac{1}{\mu_x^2} = \frac{1}{\mu_L^2} - \frac{1}{\mu_L}\frac{1}{\mu_{L-x}} \quad (b) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} U_Q = U_{dQ} + U_{Q-dQ}: \quad \frac{1}{C_Q} = \frac{1}{C_{dQ}} + \frac{1}{C_{Q-dQ}}, \frac{1}{C_{dQ}^2} = \frac{1}{C_Q^2} - \frac{1}{C_Q}\frac{1}{C_{Q-dQ}}, \quad (c) \\ Q = dQ + (Q-dQ): \quad C_Q = C_{dQ} + C_{Q-dQ}, C_{dQ}^2 = C_Q^2 - C_Q C_{Q-dQ}. \quad (d) \end{cases} \quad (2.8)$$

The proper oscillation frequencies $\omega_0^{\ 2}$ in electrodynamic systems are directly related to their elastic $\omega_0^{\ 2}=\frac{\mu}{m}$ and electrical $\omega_0^{\ 2}=\frac{1}{LC}$ properties. Therefore, from the relation (2.6) it is possible to establish the patterns of the formation of the frequency spectrum in the process of binary structuring for GP[19]. In these circles, oscillatory processes are periodic, therefore, the use of the GP method can be limited to one period T_Ω by introducing the time τ for which the model of the GP has the form:

$$\frac{\tau}{T_\Omega}=\frac{T_\Omega-\tau}{\tau}. \qquad (2.9)$$

We turn to the statistical analysis of electrodynamic systems with binary GP structuring. Binary structuring of GP systems can be investigated by an additive energy parameter. Then

$$W_\Lambda=W_\xi+W_{\Lambda-\xi}\Rightarrow\frac{W_\xi}{W_\Lambda}=\frac{W_\Lambda-W_\xi}{W_\xi}\Rightarrow W_\xi^{\ 2}=W_\Lambda^{\ 2}-W_\Lambda W_\xi\Rightarrow$$

$$\Rightarrow\left(\frac{W_\xi}{W_\Lambda}\right)^2-1-\frac{W_\xi}{W_\Lambda}-\varphi_W^{\ 2}-1-\varphi_W \quad \varphi_{W,\pm}-\frac{1\pm\sqrt{5}}{2} \qquad (2.10)$$

and the algorithm for further analysis will be determined by the scheme of the combination of elements among themselves.

2.2 Probabilistic characteristics of binary structuring for GP

According to (2.1), the regularities of the binary separation for the physical models (2.6) reflect the recurrence sequences (2.2) with the distribution (1.5). If this sequence of numbers is increasing, then the mean and variance of random values (RV) of numbers with the Fibonacci distribution, at the limit $n\to\infty$, depend on the volume[20-22]. Therefore, the author[23] proposed the probabilistic characteristics to determine for a descending sequence of type

$$Y_n=n+1-X_n, \qquad (2.11)$$

However, author[23] developed this method for the distribution (1.3) with values $p=1,q=1$.

Consider the method of calculating the mean and variance of distributions and their limit values ($n\to\infty$) of the decreasing type (2.11) of the RV sequence with a general distribution (1.5) with values $p\rangle 1, q\rangle 1$, including for the values formed on the phase plane $p0q$ of the parabola (1.7). To do this, we introduce a RV $X_n=i$ with probability

$$\theta_n(i)=\frac{G_i}{\sum_{i=1}^{N}G_i}, \quad \forall i=1,2,...,n, \qquad (2.12)$$

acquires discrete values from 1 to n. The mean RV (2.11) is calculated as:

$$\overline{Y_n}=\frac{\sum_{i=1}^{n}(n+1-i)\cdot G_i}{\sum_{i=1}^{n}G_i}=n+1-\frac{\sum_{i=1}^{n}i\cdot G_i}{\sum_{i=1}^{n}G_i} \qquad (2.13)$$

The variance is equal

$$D[Y_n]=\overline{Y_n^{\ 2}}-\left(\overline{Y_n}\right)^2=\frac{\sum_{i=1}^{n}(n+1-i)^2\cdot G_i}{\sum_{i=1}^{n}G_i}-\left(\overline{Y_n}\right)^2=(n+1)^2-2(n+1)\frac{\sum_{i=1}^{n}i\cdot G_i}{\sum_{i=1}^{n}G_i}+$$
$$+\frac{\sum_{i=1}^{n}i^2\cdot G_i}{\sum_{i=1}^{n}G_i}=(n+1)^2+2(n+1)\left(\overline{Y_n}-n-1\right)+\frac{\sum_{i=1}^{n}i^2\cdot G_i}{\sum_{i=1}^{n}G_i}-\left(\overline{Y_n}\right)^2, \qquad (2.14)$$

Table 1. Δ_{n+2}=G_{n+2}-1

$\frac{\Delta_{n+2}-(p-1)G_{n+1}}{p+q-1}$	…	$\frac{\Delta_{n+2}-(p-1)G_{n+1}}{p+4}$	$\frac{\Delta_{n+2}-(p-1)G_{n+1}}{p+3}$	$\frac{\Delta_{n+2}-(p-1)G_{n+1}}{p+2}$	$\frac{\Delta_{n+2}-(p-1)G_{n+1}}{p+1}$	$\frac{\Delta_{n+2}-(p-1)G_{n+1}}{p}$	p
…	…	…	…	…	…	…	…
$\frac{\Delta_{n+2}-3G_{n+1}}{p+q-1}$	…	$\frac{\Delta_{n+2}-3G_{n+1}}{8}$	$\frac{\Delta_{n+2}-3G_{n+1}}{7}$	$\frac{\Delta_{n+2}-3G_{n+1}}{6}$	$\frac{\Delta_{n+2}-3G_{n+1}}{5}$	$\frac{\Delta_{n+2}-3G_{n+1}}{4}$	p=4
$\frac{\Delta_{n+2}-2G_{n+1}}{q+3}$	…	$\frac{\Delta_{n+2}-2G_{n+1}}{7}$	$\frac{\Delta_{n+2}-2G_{n+1}}{6}$	$\frac{\Delta_{n+2}-2G_{n+1}}{5}$	$\frac{\Delta_{n+2}-2G_{n+1}}{4}$	$\frac{\Delta_{n+2}-2G_{n+1}}{3}$	p=3
$\frac{\Delta_{n+2}-1G_{n+1}}{q+2}$	…	$\frac{\Delta_{n+2}-1G_{n+1}}{6}$	$\frac{\Delta_{n+2}-1G_{n+1}}{5}$	$\frac{\Delta_{n+2}-1G_{n+1}}{4}$	$\frac{\Delta_{n+2}-1G_{n+1}}{3}$	$\frac{\Delta_{n+2}-1G_{n+1}}{2}$	p=2
$\frac{\Delta_{n+2}-0G_{n+1}}{q}$	…	$\frac{\Delta_{n+2}-0G_{n+1}}{5}$	$\frac{\Delta_{n+2}-0G_{n+1}}{4}$	$\frac{\Delta_{n+2}-0G_{n+1}}{3}$	$\frac{\Delta_{n+2}-0G_{n+1}}{2}$	$\frac{\Delta_{n+2}-0G_{n+1}}{1}$	p=1
q	…	q=-5	q=-4	q=-3	q=-2	q=-1	

where, according to table 1, the sum $\sum_{i=1}^{n} G_i(p,q)$ is equal to:

$$S_{G,p,q} = \sum_{i=1}^{n} G_i(p,q) = \sum_{i=1}^{n} (pG_{i-1}(p,q) + qG_{i-2}(p,q))$$
$$= \frac{G_{n+2}(p,q) - (p-1)G_{n+1}(p,q) - 1}{p+q-1}. \quad (2.15)$$

In more detail, the algorithm of computation is set out in article[24] here we only note the main points of this algorithm:

1. To calculate the sum $\sum_{i=1}^{n} i \cdot G_i$ in (2.13), by the analogy (2.15) a function is introduced

$$S(p,q,i) = \frac{G_{i+2}(p,q) - (p-1)G_{i+1}(p,q) - 1}{p+q-1}. \quad (2.16)$$

Then in limit $n \to \infty$ we get it

$$\overline{Y_n}_{\lim} = \frac{q+1+(2-p))\cdot\varphi(p,q)}{(p+q-1)\left(1-(p-1)\cdot\frac{\varphi(p,q)}{q}\right)}. \quad (2.17)$$

In a binary structured GP system

$$\begin{cases} \varphi(p,q)\cdot\dfrac{q}{\varphi(p,q)} = q, \\ \varphi(p,q) + \dfrac{q}{\varphi(p,q)} = p, \end{cases} \quad (2.18)$$

and in the limit $n \to \infty$ we have:

$$\overline{Y_n}_{\lim} = \frac{q}{q-\varphi(p,q)}. \quad (2.19)$$

2. To calculate the variance (2.14), we convert the sum $\sum_{i=1}^{n} i^2 \cdot G_{n+1-i}(p,q)$ as:

$$\sum_{i=1}^{n} i^2 \cdot G_{n+1-i}(p,q) = 1\cdot G_n + 4\cdot G_{n-1} + 9\cdot G_{n-2}$$
$$+16\cdot G_{n-3} + ... + n^2\cdot G_1 = (G_n + G_{n-1} + G_{n-2}$$
$$+ G_{n-3} + ... + G_1) + 3(G_{n-1} + G_{n-2} + G_{n-3} + ... + G_1) \quad (2.20)$$
$$+ 5(G_{n-2} + G_{n-3} + ... + G_1) + ... + (2n-1)G_1 =$$
$$\sum_{i=1}^{n} (2i-1)\cdot S_{n+1-i}(p,q)$$

Then the mean of the square $\overline{Y_n^2}$ at the $n \to \infty$ is equals

$$\overline{Y_n^2}_{\lim} = \left(2\overline{Y_n}_{\lim} - 1\right)\cdot\frac{(p-1)^2 + p^2 + q + p\varphi p(p,q) - 2(p-1)(p+\varphi p(p,q))}{(p+q-1)\left[1-(p-1)\varphi p(p,q)\right]} \quad (2.21)$$

and the final equation for variance $\left(\overline{Y_n^2}_{\lim} - \left(\overline{Y_n}_{\lim}\right)^2\right)$ in a binary structured system will have the form:

$$D_{\lim} = \overline{Y_n^2}_{\lim} - \left(\overline{Y_n}_{\lim}\right)^2 = \frac{q\cdot\varphi(p,q)}{(q-\varphi(p,q))^2}. \quad (2.22)$$

The limits for the mean (2.19) and the variance (2.22) represent a practical interest in terms of applying the gold division model of the whole electrodynamic system to two unequal parts, based on the proportional equality of the ratio of the divided parts, which satisfies the square equation with the coefficients p,q.

3 Conclusions

In this paper, based on the physical analogy between processes in elastic-dynamic and electrical systems, a mathematical model of binary structuring of the additive parameter is developed on the basis of the golden ratio of the ratio of the divided parts. It is shown that the direct consequence of the use of binary software structuring is the generation of the Fibonacci sequence. For electrodynamic systems of the Newtonian type, a method for calculating the boundary values of the mean and the dispersion of the probabilistic distribution of the Fibonacci additive parameters that are binary structured according to the golden ratio.

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