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Research Article

On Delay-dependent and Delay-derivative-dependent Stability and Stabilization for Lur'e Systems with Slow-varying Delay

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Abstract: This note is concerned with the absolute stability for time-varying delay Lur'e system with sector-bounded nonlinearity. Improved delay-dependent and delay-derivative-dependent stability criteria are obtained in the form of linear matrix inequalities (LMIs) by constructing a modified augmented Lyapunov-Krasovskii (LK) functional without applying the model transformation or the bounding techniques for cross terms. Thus, the presented delay-dependent criteria are less conservative than those in the literature. Moreover, state feedback stabilizing controllers based on the proposed stability criteria are designed. Numerical example demonstrates the effectiveness and superiority of the proposed method.

Keywords: Lur'e systems; Time-delay; Absolute stability; LMI; Stabilization

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1 Introduction

Since Lur'e and Postnikov introduced the concept of absolute stability and the Lur'e problem in 1944, absolute stability problem for typical Lur'e control systems have been extensively investigated and achieved a great number of results[1-4]. Many nonlinear systems e.g. Chua's circuit and Lorenz system can be classified into this type[5]. On the other hand, time-delays as one of main source of instability and poor performance universally exist in various dynamic systems such as engineering system, biological system, chemical system and electrical networks, *etc*. Thus, the absolute stability for Lur'e systems with time-delay is of considerable significance[6-13]. According to whether stability criteria contain time-delays information or not, they can be divided into delay-dependent ones and delay-independent ones. In general, the former is less conservative than the latter especially when delay is small[14-17].

In order to deduce delay-dependent absolute stability criteria, one usually transforms the original system to a distributed system and then uses the bounding techniques for cross terms. For example, Yu[18] employed model transformation and Moon's inequality to discuss absolute stability and stabilization for a class of nonlinear system with time delay. However, model transformation may induce additional dynamics. Recently, a method called the free-weighting matrix (FWM) approach was presented to analyze the stability and synthesize time-delayed control systems. By this method, a series of less conservative results were obtained[15-17]. Wu[7] used an augmented LK functional and FWM to study absolute stability for Lurie systems with time delay. However, the models discussed [7] andare constant delay not time-varying delay and their results are not applicable to Lur'e systems with time-varying delay [18]. Though [13, 19, 20] studied the absolute stability for Lur'e systems with time-varying delay and obtained stability criteria, it is time-consuming to test these criteria owing to many slack matrix variables containing in them[21-30]. A delay decomposition approach is introduced in[31, 32] to study delayed control systems and some less conservative criteria are obtained. Nevertheless, the results derived by the delay decomposition approach often include larger number of LMIs and matrix variables. As pointed out in [33,34], the number of LMIs increases in the order of 2^N with the number of decomposition N, which increases dramatically computational burden. Motivated by above discussion, we have this work.

In this paper, we will focus on the problem of stability for Lur'e system with time-varying delay and sector-bounded nonlinearity via a triple integral form of LK functional and our lemmas. Absolute stability criteria are deduced for Lur'e systems with time-varying delay in terms of LMIs. Based on the obtained absolute stability criteria, design of state feedback stabilizing controllers is available and the controller gains can be given by solving matrix inequality. Finally, numerical example shows the effectiveness and merit of the proposed approach.

Notations. The following notations are used throughout the paper. The superscript "T" denotes the transpose of a matrix. $R^{n\times m}$ and $R^{n\times m}$ stand for the set of real vector with n-dimensional and real matrix of size $n\times m$, respectively. $P>0\,(P\geq 0)$ symbolize a symmetric positive definite (positive semi-definite) matrix. $P<0\,(P\leq 0)$ represents a symmetric negative definite(negative semi-definite) matrix. I and 0 refer to the identity matrix and zero matrix with compatible dimensions, respectively. Block diagonal matrix is symbolized by diag(…). In a symmetric matrix, the symbol "*" is used to denote the term that is induced by symmetry, e.g. $\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$. In what follows, we always assume the matrices are of compatible dimensions with context.

2 Problem description and preliminaries

Consider the following system:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + A_1 x(t-d(t)) + Dw(t) \\ &z(t) = Mx(t) + Nx(t-d(t)) \\ &w(t) = -\varphi(t, z(t)) \\ &x(\theta) = \psi(\theta),\ \ \theta \in [-h, 0] \end{aligned} \tag{1}$$

where $x(t)\in R^n, w(t)\in R^m$ *and* $z(t)\in R^m$ are the state vector, input vector and output vector of the system, respectively; A, A_1, D, M and N are known real constant matrices of appropriate dimensions. $\psi(\cdot)$ is a continuous vector-valued initial function. $\varphi(t,z(t)):[0,+\infty)\times R^m \to R^m$ is a class of memoryless, time-varying nonlinear vector-valued function that is piecewise continuous in t and globally Lipschitz in $z(t)$, $\varphi(t,0)=0$ and satisfies the following condition for $\forall t\geq 0, \forall z(t)\in R^m$:

$$\varphi^T(t,z(t))[\varphi(t,z(t)) - Kz(t)] \leq 0 \tag{2}$$

$$[\varphi(t,z(t)) - K_1 z(t)]^T[\varphi(t,z(t)) - K_2 z(t)] \leq 0 \tag{3}$$

where K_1 and K_2 are real constant matrices with appropriate dimensions and $K= K_2-K_1$ is a symmetric positive definite matrix. The nonlinear connection function $\varphi(t,z(t)$ satisfying (2) is said to belong to the sector [0, K] symbolized by $\varphi(\cdot,\cdot)\in S[0,K]$ and $\varphi(t,z(t)$ satisfying (3), it is said to belong to the sector [K_1, K_2] symbolized by $\varphi(\cdot,\cdot)\in S[K_1,K_2]$ .The time delay $d(t)$ satisfies the following conditions:

$$0 < d(t) \leq h < +\infty \ \ and\ \ \dot{d}(t) \leq \mu < 1, \forall t \geq 0 \tag{4}$$

where h *and* μ are known real constants.

Remark 1. Due to the fact that time-varying delay often varies slowly in practical systems, the assumption $\dot{d}(t)\leq\mu<1$ is reasonable. According to [28], this type of time delay is called a slowly-varying delay.

In this paper, we investigate not only the absolute stability of nominal system (1), but also the following system with time-varying structured uncertainties:

$$\begin{aligned} &\dot{x}(t) = (A+\Delta A(t))x(t) + (A_1 + \Delta A_1(t))x(t-d(t)) + Dw(t) \\ &z(t) = Mx(t) + Nx(t-d(t)) \\ &w(t) = -\varphi(t, z(t)) \\ &x(\theta) = \psi(\theta),\ \ \theta \in [-h, 0] \end{aligned} \tag{5}$$

where the time-varying structured uncertainties with the following form

$$[\Delta A(t)\ \ \Delta A_1(t)] = LF(t)[E_a\ \ E_{a1}] \tag{6}$$

where L, E_a and E_{a1} are constant matrices with appropriate dimensions, $F(t)$ is an unknown real and possibly time-varying matrix with Lebesgue measurable elements satisfying

$$F^T(t)F(t) \leq I \qquad \forall t > 0 \tag{7}$$

Definition1. (Han [25]) The system (1) is said to be absolutely stable in the sector [K_1, K_2] if a trivial solution $x(t)$=0 is globally uniformly asymptotically stable for any nonlinear function $\varphi(t,z(t)$ satisfying (3).

Definition2. (Han [25]) The uncertain system (5) subject to (6) and (7) is said to be robustly absolutely stable in the sector [K_1, K_2] if a trivial solution $x(t)$ =0 is globally uniformly asymptotically stable for any nonlinear function $\varphi(t, z(t)$ satisfying (3).

Before moving on, we introduce the following lemmas which play an important role in deriving our main results.

Lemma 1. (Petersen I R *et al* [24]) Given matrices Σ, Ξ, and Ω with $\Omega=\Omega^T$, then

$$\Omega + \Sigma F(t)\Xi + \Xi^T F^T(t)\Sigma^T < 0$$

holds for all F(t) satisfying (7) if and only if there exists a scalar $\varepsilon > 0$ such that

$$\Omega + \varepsilon^{-1}\Sigma \quad ^T + \varepsilon \Xi^T \Xi < 0$$

Lemma 2 (Jensen's integral inequality [33]) For any constant matrix $R \in R^{n\times n}$, $R = R^T > 0$, a scalar $h > 0$ and a vector-valued function $w(t):[0,h] \to R^n$ such that the following integration are well defined , then

$$\left(\int_{t-h}^{t} w(s)ds\right)^T R\left(\int_{t-h}^{t} w(s)ds\right) \le h\int_{t-h}^{t} w^T(s)Rw(s)ds \quad (8)$$

Lemma 3 For any constant matrix $R \in R^{n\times n}$, $R = R^T > 0$, scalar $\tau_2 > \tau_1 \ge 0$, and a vector-valued function $w(t)$ such that the following integration are well defined ,then the following inequality holds

$$\begin{aligned}&(\int_{-\tau_2}^{-\tau_1} d\theta \int_{t+\theta}^{t} w^T(s)ds)R(\int_{-\tau_2}^{-\tau_1} d\theta \int_{t+\theta}^{t} w(s)ds)\\ &\le \frac{\tau_2^2 - \tau_1^2}{2}\int_{-\tau_2}^{-\tau_1} d\theta \int_{t+\theta}^{t} w^T(s)Rw(s)ds\end{aligned} \quad (9)$$

Proof. From the known condition and Schur complement, it's indubitable to find that

$$\begin{bmatrix} w^T(s)Rw(s) & w^T(s) \\ w(s) & R^{-1} \end{bmatrix} \ge 0$$

Integrating the above inequality from $t+\theta$ to t with respect to s yields

$$\begin{bmatrix} \int_{t+\theta}^{t} w^T(s)Rw(s)ds & \int_{t+\theta}^{t} w^T(s)ds \\ \int_{t+\theta}^{t} w(s)ds & -\theta R^{-1} \end{bmatrix} \ge 0$$

Then integrating the obtained inequality again from $-\tau_2$ to $-\tau_1$ with respect to θ yields

$$\begin{bmatrix} \int_{-\tau_2}^{-\tau_1} d\theta \int_{t+\theta}^{t} w^T(s)Rw(s)ds & \int_{-\tau_2}^{-\tau_1} d\theta \int_{t+\theta}^{t} w^T(s)ds \\ \int_{-\tau_2}^{-\tau_1} d\theta \int_{t+\theta}^{t} w(s)ds & \int_{-\tau_2}^{-\tau_1} -\theta R^{-1} d\theta \end{bmatrix} \ge 0$$

Utilizing Schur complement again, inequality (9) can be derived. This completes its proof.

3 Main results

For the sake of simplicity, we firstly consider the case where the nonlinear connection function $\varphi(t, z(t) \in S[0, K]$ i.e. $\varphi(t, z(t)$ satisfying (2).

Theorem 1 The system (1) satisfying (4) with $\varphi(t, z(t) \in S[0, K]$ is absolutely stable for given h and μ, if there exist real matrices

$$\begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix}^T > 0, \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix}^T > 0, \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix}^T > 0,$$

$R = R^T > 0$ such that the following LMIs holds

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) & 0 & (1,5) & (1,6) & A^T\Phi \\ * & -\frac{1}{h}Z_{22} & 0 & 0 & -P_{22} + \frac{1}{h}Z_{12} & 0 & 0 \\ * & * & (\mu-1)Q_{11} & (\mu-1)Q_{12} & A_1^T P_{12} & -N^T K^T & A_1^T\Phi \\ * & * & * & (\mu-1)Q_{22} & 0 & 0 & 0 \\ * & * & * & * & -\frac{1}{h}Z_{11} - \frac{2}{h^2}R & P_{12}^T D & 0 \\ * & * & * & * & * & -2I & D^T\Phi \\ * & * & * & * & * & * & -\Phi \end{bmatrix} < 0 \quad (10)$$

where

$$(1,1) \triangleq P_{11}A + A^T P_{11} + Q_{12}A + A^T Q_{12}^T + hZ_{12}A + hA^T Z_{12}^T + P_{12} + P_{12}^T + Q_{11} + hZ_{11} - 2R\frac{1}{h} Z_{22},$$

$$(1,2) \triangleq -P_{12} + \frac{1}{h}Z_{22},\ (1,3) \triangleq P_{11}A_1 + Q_{12}A_1 + hZ_{12}A_1,\ (1,5) \triangleq P_{22} + A^T P_{12} - \frac{1}{h}Z_{12} + \frac{2}{h}R,$$

$$(1,6) \triangleq P_{11}D + Q_{12}D + hZ_{12}D - M^T K^T,$$

$$\Phi = Q_{22} + hZ_{22} + \frac{1}{2}h^2 R.$$

Proof. Choose the following LK functional candidate as

$$\begin{aligned} V(t, x_t) &= \begin{bmatrix} x(t) \\ \int_{t-h}^{t} x(s)ds \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ \int_{t-h}^{t} x(s)ds \end{bmatrix} \\ &+ \int_{t-d(t)}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds \\ &+ \int_{-h}^{0} d\theta \int_{t+\theta}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds \\ &+ \int_{-h}^{0} d\theta \int_{\theta}^{0} d\alpha \int_{t+\alpha}^{t} \dot{x}^T(s)R\dot{x}(s)ds \end{aligned}$$

where

$$\begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix}^T > 0, \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix}^T > 0, \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix}^T > 0,$$

$R = R^T > 0$ are to be determined matrices.

The time-derivative of the LK functional along the trajectory of (1) is given by

$$
\begin{aligned}
\dot{V}(t,x_t) &= 2\begin{bmatrix} x(t) \\ \int_{t-h}^{t} x(s)ds \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ x(t)-x(t-h) \end{bmatrix} + \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
&- \begin{bmatrix} x(t-d(t)) \\ \dot{x}(t-d(t)) \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} \begin{bmatrix} x(t-d(t)) \\ \dot{x}(t-d(t)) \end{bmatrix} (1-\dot{d}(t)) + h \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
&- \int_{t-h}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds + \frac{1}{2}h^2 \dot{x}^T(t)R\dot{x}(t) - \int_{-h}^{0} d\theta \int_{\theta+t}^{t} \dot{x}^T(s)R\dot{x}(s)ds
\end{aligned} \tag{11}
$$

From (1) and (2), we get

$$
\begin{aligned}
&-2w^T(t)w(t) - 2w^T(t)KMx(t) \\
&-2w^T(t)KNx(t-d(t)) \geq 0
\end{aligned} \tag{12}
$$

Using Lemma 2, one has

$$
-\int_{t-h}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds \leq \frac{1}{h} \int_{t-h}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T ds \begin{bmatrix} -Z_{11} & -Z_{12} \\ * & -Z_{22} \end{bmatrix} \int_{t-h}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds \tag{13}
$$

Using Lemma 3, one gets

$$
\begin{aligned}
&-\int_{-h}^{0} d\theta \int_{\theta+t}^{t} \dot{x}^T(s)R\dot{x}(s)ds \leq \frac{2}{h^2} \int_{-h}^{0} d\theta \int_{t+\theta}^{t} \dot{x}^T(s)ds(-R) \int_{-h}^{0} d\theta \int_{t+\theta}^{t} \dot{x}(s)ds \\
&= -2x^T(t)Rx(t) + \frac{4}{h}x^T(t)R\int_{t-h}^{t} x(s)ds - \frac{2}{h^2}\int_{t-h}^{t} x^T(s)dsR\int_{t-h}^{t} x(s)ds
\end{aligned} \tag{14}
$$

Combining with (1), (4), (12), (13) and (14), the upper bound of $\dot{V}(t,x_t)$ can be estimated as

$\dot{V}(t,x_t) \leq \xi^T(t)\Xi\xi(t)$,

where

$$
\xi^T(t) = [x^T(t) \;\; x^T(t-h) \;\; x^T(t-d(t)) \;\; \dot{x}^T(t-d(t)) \;\; \int_{t-h}^{t} x^T(s)ds \;\; w^T(t)]
$$

$$
\Xi = \begin{bmatrix}
(1,1) & (1,2) & (1,3) & 0 & (1,5) & (1,6) \\
* & -\frac{1}{h}Z_{22} & 0 & 0 & -P_{22}+\frac{1}{h}Z_{12} & 0 \\
* & * & (3,3) & (\mu-1)Q_{12} & A_1^T P_{12} & A_1^T\Phi D - N^T K^T \\
* & * & * & (\mu-1)Q_{22} & 0 & 0 \\
* & * & * & * & -\frac{1}{h}Z_{11}-\frac{2}{h^2}R & P_{12}^T D \\
* & * & * & * & * & -2I + D^T\Phi D
\end{bmatrix}
$$

with

$$
\begin{aligned}
(1,1) &\triangleq P_{11}A + A^T P_{11} + P_{12} + P_{12}^T + A^T\Phi A + Q_{11} + hZ_{11} - 2R \\
&\quad - \frac{1}{h}Z_{22} + Q_{12}A + A^T Q_{12}^T + hZ_{12}A + hA^T Z_{12}^T, \\
(1,2) &\triangleq -P_{12} + \frac{1}{h}Z_{22}, \\
(1,3) &\triangleq P_{11}A_1 + Q_{12}A_1 + hZ_{12}A_1 + A^T\Phi A_1, \\
(1,5) &\triangleq P_{22} + A^T P_{12} - \frac{1}{h}Z_{12} + \frac{2}{h}R, \\
(1,6) &\triangleq P_{11}D + Q_{12}D + hZ_{12}D - M^T K^T + A^T\Phi D, \\
(3,3) &\triangleq (\mu-1)Q_{11} + A_1^T\Phi A_1, \\
\Phi &= Q_{22} + hZ_{22} + \frac{1}{2}h^2 R.
\end{aligned}
$$

According to the Lyapunov-Krasovskii stability theory, system (1) is absolutely stable if $\Xi < 0$ holds. In view of Schur complement, the inequality $\Xi < 0$ is equivalent to (10). This completes the proof.

For the nonlinearity $\varphi(t,z(t)$ satisfying the general sector condition (3), that is $\varphi(t,z(t) \in S[K_1,K_2]$, applying loop transformation [2], we can conclude that absolute stability of system (1) in the sector $[K_1,K_2]$ is equivalent to that of the following system in the sector $[0,K_2-K_1]$:

$$
\begin{aligned}
\dot{x}(t) &= (A-DK_1M)x(t) + (A_1-DK_1N)x(t-d(t)) + Dw(t) \\
z(t) &= Mx(t) + Nx(t-d(t)) \\
w(t) &= -\varphi(t,z(t))
\end{aligned} \tag{15}
$$

Thus, we can easily derive Theorem 2 from Theorem 1.

Theorem 2 The system (1) satisfying (4) with $\varphi(t,z(t) \in S[K_1,K_2]$ is absolutely stable for given h and μ, if there exist real matrices

$$
\begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix}^T > 0, \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix}^T > 0, \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix}^T > 0,
$$

$R = R^T > 0$ such that $\Omega < 0$.

where

$$
\Omega = \begin{bmatrix}
(1,1) & (1,2) & (1,3) & 0 & (1,5) & (1,6) & (A-DK_1M)^T\Phi \\
* & -\frac{1}{h}Z_{22} & 0 & 0 & (2,5) & 0 & 0 \\
* & * & (\mu-1)Q_{11} & (\mu-1)Q_{12} & (3,5) & (3,6) & (A_1-DK_1N)^T\Phi \\
* & * & * & (\mu-1)Q_{22} & 0 & 0 & 0 \\
* & * & * & * & (5,5) & P_{12}^T D & 0 \\
* & * & * & * & * & -2I & D^T\Phi \\
* & * & * & * & * & * & -\Phi
\end{bmatrix} \tag{16}
$$

with

$$
\begin{aligned}
(1,1) &\triangleq P_{11}(A-DK_1M) + (A-DK_1M)^T P_{11} \\
&\quad + P_{12} + P_{12}^T + Q_{11} + hZ_{11} - 2R - \frac{1}{h}Z_{22} \\
&\quad + Q_{12}(A-DK_1M) + (A-DK_1M)^T Q_{12}^T \\
&\quad + hZ_{12}(A-DK_1M) + h(A-DK_1M)^T Z_{12}^T, \\
(1,2) &\triangleq -P_{12} + \frac{1}{h}Z_{22}, \; (1,3) \triangleq P_{11}(A_1-DK_1N) \\
&\quad + Q_{12}(A_1-DK_1N) + hZ_{12}(A_1-DK_1N), \\
(1,5) &\triangleq P_{22} + (A-DK_1M)^T P_{12} - \frac{1}{h}Z_{12} + \frac{2}{h}R, \\
(1,6) &\triangleq P_{11}D + Q_{12}D + hZ_{12}D - M^T(K_2-K_1)^T, \\
(2,5) &\triangleq -P_{22} + \frac{1}{h}Z_{12}, \\
(3,5) &\triangleq (A_1-DK_1N)^T P_{12}, \; (3,6) \triangleq \\
&\quad -N^T(K_2-K_1)^T, \; (5,5) \triangleq -\frac{1}{h}Z_{11} - \frac{2}{h^2}R.
\end{aligned}
$$

In addition, Φ is defined in Theorem 1.

Remark 2 In previous works e.g. [14, 21-23], the term $-\int_{t-h}^{t} \dot{x}^T(s)Z\dot{x}(s)ds$ was enlarged as. $-\int_{t-d(t)}^{t} \dot{x}^T(s)Z\dot{x}(s)ds$ Note that

$$
-\int_{t-h}^{t} \dot{x}^T(s)Z\dot{x}(s)ds = -\int_{t-d(t)}^{t} \dot{x}^T(s)Z\dot{x}(s)ds - \int_{t-h}^{t-d(t)} \dot{x}^T(s)Z\dot{x}(s)ds
$$

Clearly, the term $-\int_{t-h}^{t-d(t)} \dot{x}^T(s)Z\dot{x}(s)ds$ was ignored in estimating the derivative of $V(t,x_t)$, which may result in conservativeness. In this paper, we overcome this deficiency.

Remark 3 Theorem 1 is derived from neither model transformation nor the bounding techniques for cross terms of derivative of the Lyapunov functional. Therefore, none of the additional dynamics is induced in this stability criterion.

Remark 4 Less conservatism of the proposed stability criteria is attributed to two aspects: the first is that we constructed augmented Lyapunov-Krasovskii functional with a triple integral term where the information on the upper bound of the delay and its derivative are under full consideration; the second has been pointed out in Remark 2.

Now we turn to discuss the system (5) for robustly absolutely stable based on Theorem 2. Under the condition of (6), the system (5) can be converted into the following system:

$$\begin{aligned}
&\dot{x}(t) = (A + LF(t)E_a)x(t) + (A_1 + LF(t)E_{a1})x(t-d(t)) + Dw(t)\\
&z(t) = Mx(t) + Nx(t-d(t))\\
&w(t) = -\varphi(t, z(t))\\
&x(\theta) = \psi(\theta),\ \ \theta \in [-h, 0]
\end{aligned}$$

Supplanting A, A_1 in (16) by $A+E\ (t)E_a$, $A_1 + E\ (t)E_{a1}$ respectively, then we can conclude the following inequality

$$\Omega + \begin{bmatrix} \Pi \\ 0 \\ 0 \\ 0 \\ P_{12}^T L \\ 0 \\ \Phi L \end{bmatrix} F(t)\begin{bmatrix} E_a & 0 & E_{a1} & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} E_a^T \\ 0 \\ E_{a1}^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} F^T(t)\begin{bmatrix} \Pi^T & 0 & 0 & 0 & L^T P_{12} & 0 & L^T\Phi \end{bmatrix} < 0 \tag{17}$$

where

$\Pi = P_{11}L + Q_{12}L + hZ_{12}L$

In light of Lemma 1, inequality (17) holds if and only if there exists a scalar $\lambda > 0$, such that

$$\Omega + \lambda^{-1}\begin{bmatrix} \Pi \\ 0 \\ 0 \\ 0 \\ P_{12}^T L \\ 0 \\ \Phi L \end{bmatrix}\begin{bmatrix} \Pi^T & 0 & 0 & 0 & L^T P_{12} & 0 & L^T\Phi \end{bmatrix} + \lambda\begin{bmatrix} E_a^T \\ 0 \\ E_{a1}^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} E_a & 0 & E_{a1} & 0 & 0 & 0 & 0 \end{bmatrix} < 0 \tag{18}$$

From Schur complement, (18) is equivalent to (19).

$$\begin{bmatrix}
(1,1) & (1,2) & (1,3) & 0 & (1,5) & (1,6) & (1,7) & \Pi \\
* & -\frac{1}{h}Z_{22} & 0 & 0 & -P_{22} + \frac{Z_{12}}{h} & 0 & 0 & 0 \\
* & * & (3,3) & (\mu-1)Q_{12} & (3,5) & (3,6) & (3,7) & 0 \\
* & * & * & (\mu-1)Q_{22} & 0 & 0 & 0 & 0 \\
* & * & * & * & (5,5) & P_{12}^T D & 0 & P_{12}^T L \\
* & * & * & * & * & -2I & D^T\Phi & 0 \\
* & * & * & * & * & * & -\Phi & \Phi L \\
* & * & * & * & * & * & * & -\lambda I
\end{bmatrix} < 0 \tag{19}$$

where

$$\begin{aligned}
(1,1) \triangleq\ & P_{11}(A - DK_1M) + (A - DK_1M)^T P_{11} + P_{12} + P_{12}^T + Q_{11} + hZ_{11} - 2R \\
& -\frac{1}{h}Z_{22} + Q_{12}(A - DK_1M) + (A - DK_1M)^T Q_{12}^T + hZ_{12}(A - DK_1M) \\
& + h(A - DK_1M)^T Z_{12}^T + \lambda E_a^T E_a,
\end{aligned}$$

$(1,2) \triangleq -P_{12} + \frac{1}{h}Z_{22}$, $(1,5) \triangleq P_{22} + (A - DK_1M)^T P_{12} - \frac{1}{h}Z_{12} + \frac{2}{h}R$,

$(1,3) \triangleq P_{11}(A_1 - DK_1N) + Q_{12}(A_1 - DK_1N) + hZ_{12}(A_1 - DK_1N) + \lambda E_a^T E_d$,

$(1,6) \triangleq P_{11}D + Q_{12}D + hZ_{12}D - M^T(K_2 - K_1)^T$, $(1,7) \triangleq (A - DK_1M)^T\Phi$,

$(3,3) \triangleq (\mu - 1)Q_{11} + \lambda E_{a1}^T E_{a1}$, $(3,5) \triangleq (A_1 - DK_1N)^T P_{12}$, $(3,6) \triangleq -N^T(K_2 - K_1)^T$,

$(3,7) \triangleq (A_1 - DK_1N)^T\Phi$, $(5,5) \triangleq -\frac{1}{h}Z_{11} - \frac{2}{h^2}R$.

Thus, we obtain the following result.

Theorem 3 The system (5) satisfying (4) with time-varying structured uncertainties (6) is robustly absolutely stable in the sector $[K_1, K_2]$ for given h and μ, if there exist real matrices

$$\begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix}^T > 0, \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix}^T > 0, \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix}^T > 0,$$

$R = R^T > 0$ and a scalar $\lambda > 0$ such that the matrix inequality (19) holds.

On basis of Theorem 2, state feedback stabilizing controllers can be available. Discuss the following closed-loop system:

$$\dot{x}(t) = Ax(t) + A_1 x(t - d(t)) + Dw(t) + Bu(t)$$
$$z(t) = Mx(t) + Nx(t - d(t))$$
$$w(t) = -\varphi(t, z(t)) \tag{20}$$

where $u(t) \in R^p$ is the controlled input vector, B is a real constant matrix with appropriate dimensions. The nonlinear function $\varphi(t, z(t)$ satisfying (3), i.e. $\varphi(t, z(t) \in S[K_1, K_2]$.

Suppose

$$u(t) = \overline{K}x(t) \tag{21}$$

where $\overline{K} \in R^{p \times n}$ is a to be determined controller gain matrix.

Substituting the state feedback controller (21) into closed-loop system (20) yields

$$\dot{x}(t) = (A + B\overline{K})x(t) + A_1 x(t - d(t)) + Dw(t)$$

Replacing A by $A + B\overline{K}$ in (16), one obtains a new matrix inequality. After that, we pre- and post-multiply both sides of the newly-obtained inequality by

$$diag(P_{11}^{-1} \quad P_{11}^{-1} \quad P_{11}^{-1} \quad P_{11}^{-1} \quad P_{11}^{-1} \quad I \quad P_{11}^{-1})$$

and its transpose .

For simplicity, we introduce notations $X := P_{11}^{-1}, \hat{K} := \overline{K}P_1^{-1}, (\hat{\cdot}) := P_{11}^{-1}(\cdot)P_1^{-1}$, then we get

$$\Sigma + \Lambda X^{-1}\Gamma^T + \Gamma X^{-1}\Lambda^T < 0 \tag{22}$$

where

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & 0 & \Sigma_{15} & \Sigma_{16} & 0 \\ * & -\frac{1}{h}\hat{Z}_{22} & 0 & 0 & -\hat{P}_{22} + \frac{\hat{Z}_{12}}{h} & 0 & 0 \\ * & * & (\mu-1)\hat{Q}_{11} & (\mu-1)\hat{Q}_{12} & 0 & \Sigma_{36} & 0 \\ * & * & * & (\mu-1)\hat{Q}_{22} & 0 & 0 & 0 \\ * & * & * & * & \Sigma_{55} & 0 & 0 \\ * & * & * & * & * & -2I & 0 \\ * & * & * & * & * & * & \Sigma_{77} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} (\hat{Q}_{12} + h\hat{Z}_{12})^T & 0 & 0 & 0 & \hat{P}_{12} & 0 & \hat{Q}_{22} + h\hat{Z}_{22} + \frac{1}{2}h^2\hat{R} \end{bmatrix}^T,$$

$$\Gamma^T = \begin{bmatrix} AX + B\hat{K} - DK_1MX & 0 & A_1X - DK_1NX & 0 & 0 & D & 0 \end{bmatrix},$$

with

$$\Sigma_{11} = (AX + B\hat{K} - DK_1MX) + (AX + B\hat{K} - DK_1MX)^T + \hat{P}_{12} + \hat{P}_{12}^T + \hat{Q}_{11} + h\hat{Z}_{11} - 2\hat{R} - \frac{1}{h}\hat{Z}_{22},$$

$$\Sigma_{12} = -\hat{P}_{12} + \frac{1}{h}\hat{Z}_{22}, \quad \Sigma_{13} = A_1X - DK_1NX,$$

$$\Sigma_{15} = \hat{P}_{22} - \frac{1}{h}\hat{Z}_{12} + \frac{2}{h}\hat{R},$$

$$\Sigma_{16} = D - XM^T(K_2 - K_1)^T,$$

$$\Sigma_{36} = -XN^T(K_2 - K_1)^T,$$

$$\Sigma_{55} = -\frac{1}{h}\hat{Z}_{11} - \frac{2}{h^2}\hat{R}, \ \Sigma_{77} = -\hat{Q}_{22} - h\hat{Z}_{22} - \frac{1}{2}h^2\hat{R}.$$

In fact, the following inequality always holds

$$\Lambda X^{-1}\Gamma^T + \Gamma X^{-1}\Lambda^T \le \Lambda J^{-1}\Lambda^T + \Gamma X^{-1}JX^{-1}\Gamma^T \tag{23}$$

for any $J = J^T > 0$.

Substituting (23) into (22) and applying Schur complement yields

$$\begin{bmatrix} \Sigma & \Lambda & \Gamma \\ * & -J & 0 \\ * & * & -XJ^{-1}X \end{bmatrix} < 0 \tag{24}$$

In view of $X := P_{11}^{-1}, \hat{K} := \overline{K}P_{11}^{-1}$, the memory less state feedback control law can be expressed as $u = \hat{K}X^{-1}x(t)$.

To summarize up the aforementioned, we achieve Theorem 4.

Theorem 4 For given μ and h, closed-loop system (20) satisfying (4) with $\varphi(t, z(t) \in S[K_1, K_2]$ is absolutely stable if there exist

$$\hat{P} = \begin{bmatrix} X & \hat{P}_{12} \\ * & \hat{P}_{22} \end{bmatrix} = \begin{bmatrix} X & \hat{P}_{12} \\ * & \hat{P}_{22} \end{bmatrix}^T > 0, \hat{Q} = \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ * & \hat{Q}_{22} \end{bmatrix} = \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ * & \hat{Q}_{22} \end{bmatrix}^T > 0, \hat{Z} = \begin{bmatrix} \hat{Z}_{11} & \hat{Z}_{12} \\ * & \hat{Z}_{22} \end{bmatrix} = \begin{bmatrix} \hat{Z}_{11} & \hat{Z}_{12} \\ * & \hat{Z}_{22} \end{bmatrix}^T > 0,$$

$\hat{R} = \hat{R}^T > 0$, $J = J^T > 0$ and real constant matrix $\hat{K}$ with appropriate dimensions such that (24) holds. Moreover, a memoryless state feedback control law is given by $u = \hat{K}X^{-1}x(t)$

Remark 5 It should be pointed out that (24) in Theorem 4 is no longer LMI due to the existence of nonlinear term $-XJ^{-1}X$, and a state feedback controllers gain matrix "*K*" cannot be solved directly by a convex optimization algorithm. In order to solve "*K*", researchers have found some effective approaches. There are two simple methods. One is to set $X = \lambda J$, where λ is a tuning parameter. In this case, (24) can be converted into

$$\begin{bmatrix} \Sigma & \Lambda & \Gamma \\ * & -\lambda^{-1}X & 0 \\ * & * & -\lambda X \end{bmatrix} < 0 \tag{25}$$

The other is to use the relation $XJ^{-1}X \ge 2X - J$ by noting the fact that the following inequality

holds for any $X = X^T$ and $J = J^T > 0$ with appropriate dimensions.

Then inequality (26) can guarantee (24) holds

$$\begin{bmatrix} \Sigma & \Lambda & \Gamma \\ * & -J & 0 \\ * & * & J - 2X \end{bmatrix} < 0 \tag{26}$$

Although inequality (25),(26) are Linear Matrix

Inequality, such treatment will lead to considerable conservatism.

In the literature [26, 27] and [29], an iterative algorithm was presented to solve this problem and obtain a suboptimal solution. For detail, see ([26], [27], [29] and [30] *etc*). For reason of space, one omits it here.

Remark 6 Similar to Theorem 4, we can also obtain state feedback controllers to robustly stabilize for system (5) via Theorem 3.

4 Numerical examples

In this section, we provide illustrative example to demonstrate the effectiveness of the proposed method.

Example 1 Consider the system described by (5), (3), (4), (6) and (7) with the following parameters:

$$A=\begin{bmatrix}-2 & 0\\ 0 & -0.9\end{bmatrix}, A_1=\begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix},$$

$$D=\begin{bmatrix}-0.2\\ -0.3\end{bmatrix}, K_1=0.2, K_2=0.5,$$

$$M=[0.3\quad 0.1], N=[0.1\quad 0.2],$$

$$L=\begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix}, E_a=E_{a1}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

The maximum allowable delay bound (MADB) provided by Theorem 3 in this paper for Example 1 is listed in Table 1 for different values of μ.

Table 1. Maximum allowable delay bounds h for various μ

μ	0.00	0.30	0.60	0.90
[25]	3.3056	2.0787	1.4195	0.9228
[19]	3.3056	2.2262	1.7409	1.4682
[13]	4.1077	2.4335	1.8718	1.7077
Theorem 3	4.1077	2.5280	1.9245	1.7302

From Table 1, one can see the proposed robust stability criterion has less conservatism than those in the existing literature.

5 Conclusion

The problem of absolute stability for a class of nonlinear systems has been addressed and absolute stability criteria with the general sector condition have been obtained. In order to reduce the conservatism of the criteria, a modified augmented LK functional was constructed and neither model transformation nor the bounding techniques for cross terms were used. Based on the improved absolute stability criteria, a feedback stabilizing controllers have been designed. Finally, numerical example verifies the effectiveness of the proposed criteria.

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