Journal of Contemporary Educational Research, 2024, Volume 8, Issue 12
http://ojs.bbwpublisher.com/index.php/JCER
ISSN Online: 2208-8474
ISSN Print: 2208-8466

Rediscovering Determinant Expansion on Row or Column Theorem Through Plausible Reasoning

Haiyan Liu*

College of Jincheng, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, Jiangsu Province, China

***Corresponding author:** Haiyan Liu, ziluolan7812@sina.com

Abstract: Plausible reasoning is an important approach to reasoning conclusions. In order to cultivate students' habits and abilities to use plausible reasoning, we should give the students a chance to imitate and practice plausible reasoning in our teaching. For our linear algebra course, most of the definitions and theorems in popular linear algebra textbooks are given directly. Thus, we give a concrete process that rediscovers determinant expansion on row or column theorem through plausible reasoning during our teaching to give the students the chance to learn the reasoning.

Keywords: Determinant expansion on row or column theorem; Plausible reasoning

Online publication: December 31, 2024

1. Introduction

Reasoning is divided into argumentative reasoning and plausible reasoning. Argumentative reasoning cannot generate new knowledge about the essence of our world while plausible reasoning can [1,2]. Most of the definitions and theorems in popular linear algebra textbooks are given directly [3-5]. Thus, students lack the opportunity to study plausible reasoning. Using plausible reasoning is a practical skill that needs imitation and practice so it is important for the teacher to demonstrate the method during the class.

One of the knowledge of linear algebra—determinant expansion on row or column theorem—is selected. We give the concrete teaching process to rediscover determinant expansion on row or column theorem through plausible reasoning to give students the chance to learn the reasoning.

2. A speculation of the relationship between a higher-order determinant and a lower-order determinant

In this section, we will show how to find a simple way to calculate a determinant by expanding the determinant to a lower determinant.

The expansion of the determinant is very complex when the determinant has high order. We can convert a determinant to a simpler determinant. For example, we can convert it to an upper triangle determinant. Now we study other ways to convert a determinant to a simpler determinant. We know the second-order determinant is simpler compared with the third-order determinant. We can see the determinant with lower order is simpler compared with higher order. Thus, we can convert a determinant to another determinant with a lower order. Next, we will explore how to convert the determinant.

We can observe third-order determinants and second-order determinants to try to find the relation between the two determinants. The sign of determinant is abstract. We can observe their expansion.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \quad (1)$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \quad (2)$$

$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}$ From formulas (1) and (2), the expansion of the second-order determinant has two terms and the relationship of the two terms is subtraction. Every term is a product by two elements. The expansion of the third-order determinant has six terms and the relationship of the terms is addition or subtraction, and its every term is a product by three elements. In order to relate the third-order determinant to the second-order determinant, we can choose two terms of the third-order determinant with the relationship of subtraction and having the same factor. For instance,

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} \quad (3)$$

We can convert formula (3) to formula (4) shown as follows:

$$a_{11}(a_{22}a_{33} - a_{11}a_{23}) \quad (4)$$

Formula (4) contains an expansion of a second-order determinant, that is $a_{22}a_{33} - a_{11}a_{23}$. We can dispose other four terms of third-order determinants with the same way, then we can get:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) + a_{12}\left(a_{23}a_{31} - a_{21}a_{33}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right) \quad (5)$$

We express every expansion of the second-order determinant that appeared in formula (5) with the sign of the determinant.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad (6)$$

Now, we already use some second-order determinants to express a third-order determinant. From (6) we can see a_{11}, a_{12}, a_{13} are the elements of the first rank of the third-order determinant. We can see all the elements in the second-order determinant are the elements of the third-order determinant too. We can label all the elements of the right end of formula (6) in **Figure 1**.

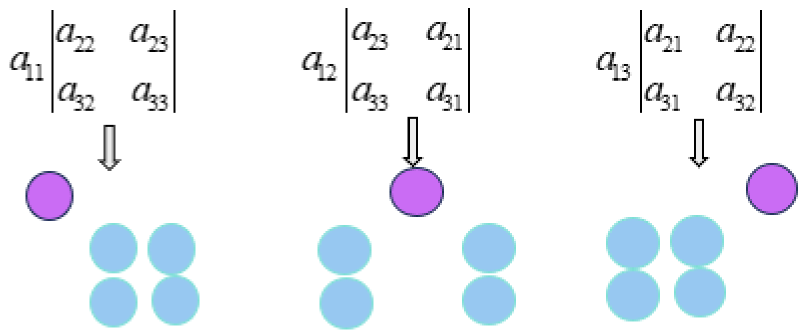

Figure 1. To label all the elements of the right end of formula (6)

From **Figure 1**, we can see the elements of the first and the third second-order determinant at the right end of formula (6) are all the elements of the third-order determinant labeled with blue circles. They form second-order determinants without changing the sequence in third-order determinants. The elements of the second second-order determinant at the right end of formula (6) are the elements of the third-order determinant, but the two columns of the elements labeled with blue circles are exchanged with each other. For the second second-order determinant, we hope that the elements labeled with a blue circle in the third-order determinant form a determinant with the same rule as the first and the third second determinant. So we represent formula (6) by formula (7):

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad (7)$$

Next, for the right end of formula (7), we will find the relationship of a_{1j} (j=1,2,3) and the second-order determinant followed by it. From **Figure 2**, we can see their elements are reminders of a_{1j}, that is, we mean the elements left when deleting the first rank and the j column. Then we can give a definition of the second-order determinant that occurred in formula (7). We name it as the complement of the element a_{1j} and denote it by M_{1j} (j=1,2,3).

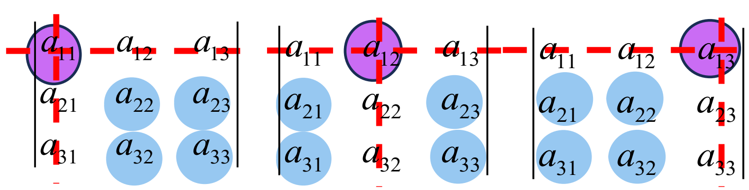

Figure 2. The relationship of a_{1j} (j=1,2,3) and the second-order determinant followed by it at the right end of formula (7)

So we can represent formula (7) by formula (8):

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \quad (8)$$

By this point, we already know the terms at right end of formula (8). Without considering the sign '+' or '-,' they are just the elements of the first rank of the third-order determinant and the complements of every element in the first rank. Next, we will study the sign before the terms. For different element a_{1j} (j=1,2,3), the sign is different. The sign is decided by the number of ranks and columns. We display the sign and the number of ranks and columns of every term at the right end of formula (8)as follows:

'+: 1,1 or 1,3' and ' - : 1,2'

From above, we can guess the relationship between the sign and the number of ranks or columns is $(-1)^{1+j}$ or $(-1)^{ij-1}$ or $(-1)^{i+j}$. Where i stands for the number of rank and j stands for the number of column. We can see, when i = 1 and j = 3, $(-1)^{1+3} = 1$ or $(-1)^{1\times3-1} = 1$ or $(-1)^{1+3} = 1$, the sign before $a_{13}M_{13}$ is "+" indeed. However, we need more evidence on which of the presentation $(-1)^{j+1}$; $(-1)^{ij-1}$; $(-1)^{i+j}$ we should choose.

From formula (8), we can see that the third-order determinant can expand by the elements in the first rank. The other rank has the same status compared with the first rank and the column has the same status compared with rank. So we can expand the third-order determinant by the same way which can get formula (8). So we can get formulas (9) and (10) as follows:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad (9)$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \quad (10)$$

$$= a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

From formulas (9) and (10), we can see the second-order determinant following the element a_{ij} is still the complement of a_{ij}, and only $(-1)^{i+j}$ can correspond the sign before $a_{ij}M_{ij}$.

So far, we can expand the third-order determinant into the forms as follows:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{i1}(-1)^{i+1}M_{i1} + a_{i2}(-1)^{i+2}M_{i2} + a_{i3}(-1)^{i+3}M_{i3}\left(i = 1,2,3\right) \quad (11)$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{1j}(-1)^{1+j}M_{1j} + a_{2j}(-1)^{2+j}M_{2j} + a_{3j}(-1)^{3+j}M_{3j}\left(j = 1,2,3\right) \quad (12)$$

From formulas (11) and (12), we can define $(-1)^{i+j}$ M_{ij} as the algebraic complement of the element a_{ij} (i,j=12,3) and $(-1)^{i+j}$ M_{ij} can be denoted by A_{ij}. Then we can represent formulas (11) and (12) as below.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3}\left(i = 1,2,3\right) \quad (13)$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j}\left(j = 1,2,3\right) \quad (14)$$

From the expansion of the third-order determinant in formulas (13) and (14), we can generalize the expansion of the *n*th-order determinant as follows:

$$D_n = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}\left(i = 1, 2, \cdots, n\right) \qquad (15)$$

$$D_n = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}\left(j = 1, 2, \cdots, n\right) \qquad (16)$$

Where $D_n = \det(a_{ij})$.

So far, we guess the expansion of a *n*th-order determinant which use the lower-order determinant to present the higher-order determinant by formulas (15) and (16). We will give the proof for our speculation showed in formulas (15) and (16) in the next section.

3. The proof of the speculation

In this section, we will give our proof. We will prove it by the definition of determinant.

The proof is shown as follows.

$$\begin{aligned}
D_n &= \sum_{(p_1p_2\cdots p_i\cdots p_n)} (-1)^{\sigma(p_1p_2\cdots p_{i-1}p_ip_{i+1}p_n)} a_{1p_1}a_{2p_2}\cdots a_{i-1,\,p_{i-1}}a_{ip_i}a_{i+1,\,p_{i+1}}\cdots a_{np_n} \\
&= \sum_{(p_1p_2\cdots 1\cdots p_n)} (-1)^{\sigma(p_1p_2\cdots p_{i-1}1p_{i+1}p_n)} a_{1p_1}a_{2p_2}\cdots a_{i-1,\,p_{i-1}}a_{i1}a_{i+1,\,p_{i+1}}\cdots a_{np_n} + \sum_{(p_1p_2\cdots 2\cdots p_n)} (-1)^{\sigma(p_1p_2\cdots p_{i-1}2p_{i+1}p_n)} a_{1p_1}a_{2p_2}\cdots a_{i-1,\,p_{i-1}}a_{i2}a_{i+1,\,p_{i+1}}\cdots a_{np_n} \\
&+\cdots+ \sum_{(p_1p_2\cdots j\cdots p_n)} (-1)^{\sigma(p_1p_2\cdots p_{i-1}jp_{i+1}p_n)} a_{1p_1}a_{2p_2}\cdots a_{i-1,\,p_{i-1}}a_{ij}a_{i+1,\,p_{i+1}}\cdots a_{np_n} +\cdots+ \sum_{(p_1p_2\cdots n\cdots p_n)} (-1)^{\sigma(p_1p_2\cdots p_{i-1}np_{i+1}p_n)} a_{1p_1}a_{2p_2}\cdots a_{i-1,\,p_{i-1}}a_{in}a_{i+1,\,p_{i+1}}\cdots a_{np_n}
\end{aligned} \qquad (17)$$

We continue to study the the *j*-th item at the right end of formula (17).

$$\begin{aligned}
&\sum_{(p_1p_2\cdots j\cdots p_n)} (-1)^{\sigma(p_1p_2\cdots p_{i-1}jp_{i+1}p_n)} a_{1p_1}a_{2p_2}\cdots a_{i-1,\,p_{i-1}}a_{ij}a_{i+1,\,p_{i+1}}\cdots a_{np_n} \\
&= a_{ij} \sum_{(p_1p_2\cdots j\cdots p_n)} (-1)^{\sigma(p_1p_2\cdots p_{i-1}jp_{i+1}p_n)} a_{1p_1}a_{2p_2}\cdots a_{i-1,\,p_{i-1}}a_{i+1,\,p_{i+1}}\cdots a_{np_n} \\
&= (-1)^{i-1}a_{ij} \sum_{(p_1p_2\cdots j\cdots p_n)} (-1)^{\sigma(jp_1p_2\cdots p_{i-1}p_{i+1}p_n)} a_{1p_1}a_{2p_2}\cdots a_{i-1,\,p_{i-1}}a_{i+1,\,p_{i+1}}\cdots a_{np_n} \\
&= (-1)^{i-1}a_{ij} \sum_{(p_1p_2\cdots p_{i-1}p_{i+1}\cdots p_n)} (-1)^{j-1}(-1)^{\sigma(p_1p_2\cdots p_{i-1}p_{i+1}p_n)} a_{1p_1}a_{2p_2}\cdots a_{i-1,\,p_{i-1}}a_{i+1,\,p_{i+1}}\cdots a_{np_n} \\
&= a_{ij}\times(-1)^{i+j} \sum_{(p_1p_2\cdots p_{i-1}p_{i+1}\cdots p_n)} (-1)^{\sigma(p_1p_2\cdots p_{i-1}p_{i+1}p_n)} a_{1p_1}a_{2p_2}\cdots a_{i-1,\,p_{i-1}}a_{i+1,\,p_{i+1}}\cdots a_{np_n}
\end{aligned} \qquad (18)$$

We can see the item which is in the last equation of formula (18) is just the algebraic complement A_{ij} of the element a_{ij}:

$$a_{ij}\times(-1)^{i+j} \sum_{(p_1p_2\cdots p_{i-1}p_{i+1}\cdots p_n)} (-1)^{\sigma(p_1p_2\cdots p_{i-1}p_{i+1}p_n)} a_{1p_1}a_{2p_2}\cdots a_{i-1,\,p_{i-1}}a_{i+1,\,p_{i+1}}\cdots a_{np_n}$$

So far, we testify our speculation shown in formula (15). We can prove formula (16) in the same way. Now, we can call the speculation as determinant expansion on row or column theorem.

4. Conclusion

Plausible reasoning is the right way to discover new knowledge. We hope to cultivate this ability to students in the concrete teaching process. For the linear algebra course, we chose the knowledge called determinant expansion on row or column theorem to show the concrete process to discover the knowledge through plausible reasoning methods. From the process, we can find that the definition is given according to our need when we get new things instead of being given directly. The conclusion is given by our observation, speculation, and proof.

The process of speculation and proof for determinant expansion on row or column theorem are both our innovative works. Our follow-up work is to write a book to show more discovery processes for the knowledge in linear algebra textbooks.

Disclosure statement

The author declares no conflict of interest.

References

[1] Polya G, 2014, Maths and Speculation, Science Press, Beijing, China.

[2] Xu LZ, 2001, On Mathematical Methodology, Shandong Education Press, Jinan, China.

[3] School of Mathematical Sciences, Tongji University, 2023, Linear Algebra, Higher Education Press, Beijing, China.

[4] Lay DC, Lay SR, 2023, Linear Algebra and Its Application, China Machine Press, Beijing, China.

[5] Tian CH, Sun YB, 2022, Linear Algebra, People's Posts and Telecommunications Press, Beijing, China.