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Vanishing Theorems for *p*-Harmonic Forms on Submanifolds in Spheres

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Abstract: In this paper, we present some vanishing theorems for *p*-harmonic forms on -super stable complete submanifold M immersed in sphere S^{n+m}. When $2 \leq 1 \leq n\text{-}2$, M has a flat normal bundle. Assuming that M is a minimal submanifold and $\delta > 1(n-1)p^2 / 4n[p-1+(p-1)^2 k_p]$, we prove a vanishing theorem for *p*-harmonic ℓ-forms.

Keywords: *p*-harmonic forms; Vanishing theorems; Submanifolds

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1. Introduction

Studying the relationship between the topological properties and geometric structure of submanifolds in various ambient spaces is an interesting problem in submanifold geometry. For instance, Bernstein theorem proves that any complete minimal graph in the Euclidean space R^3 is a plane. In R^3, do Carmo and Peng [1] and Fischer-Colbrue and Schoen [2] have demonstrated that the complete oriented stable minimal surface is a plane.

Cao *et al.* [3] proved a complete noncompact oriented stable minimal hypersurface M^n $(n \geq 3)$ in R^{n+1} must have only one end. Their proof mainly uses the Liouville theorem for harmonic maps according to Schoen and Yau [4]. So it is an interesting problem in geometry and topology to study vanishing theorems of harmonic forms on submanifolds in various ambient spaces.

There are various vanishing theorems for L^2 harmonic forms on complete submanifolds by assuming various geometric and analytic conditions. For example, Palmer [5] proved the vanishing theorem on a complete stable minimal hypersurface in R^{n+1}. Cavalcante *et al.* [6] showed some finiteness and vanishing theorems on submanifolds in a nonpositive curved pinching manifold with some conditions about the first eigenvalues and the total curvature. Furthermore, when the ambient manifold is a sphere, there are many interesting results. On a complete noncompact stable minimal hyperface M in sphere S^{n+1} $(n \leq 4)$, Zhu [7] proved that there is nontrivial L^2 harmonic form. He extended Tanno's result [8]. Zhu and Gan [9] studied a complete noncompact minimal submanifold in sphere. When $\|A\|^n_{L(M)}$ is less than a suitable constant, there is no nontrivial L^2 harmonic ℓ-form on M. On -super stable complete noncompact minimal submanifolds in S^{n+m} with $\delta > (\frac{m-1}{m})^2$, Han [10] showed that there is nontrivial L^2 harmonic ℓ-forms.

For general p-harmonic forms, Zhang [11] proved that there is no nontrivial L^q (q > 0) p-harmonic ℓ-form on a complete manifold with nonnegative Ricci curvature. Inspired by Zhang's results, Chang *et al.* [12] obtained the compactness for any bounded set of p-harmonic ℓ-forms. When the total curvature is limited by a constant, Han [13] obtained the vanishing theorem in the $L^{2\beta}$ p-harmonic ℓ-form on M. On complete noncompact submanifold M^n in a sphere S^{n+m} and M^n has flat normal bundle, when the total curvature has a suitable bound, Han [14] obtained that there are no nontrivial L^p p-harmonic ℓ-forms $2 \leq 1 \leq n-2$ on M^n.

Now we have defined the L^p p-harmonic ℓ-form space on M as follows:

$$H^{l,p}(L^p(M)) = \left\{\omega \in \Omega^l(M) \setminus d\omega = 0, \delta(\setminus \omega \setminus^{p-2}\omega) = 0, \int_M \setminus \omega \setminus^p dv <+\infty\right\}.$$

In this paper, we will prove the following theorems.

Theorem 1.1 *Let* M^n (n ≥ 3) *be a complete noncompact oriented submanifold in an (n+m)-dimensional sphere* S^{n+m}. *Assume that* M^n *is a minimal submanifold on* S^{n+m} *and* $\lambda_1(+ \delta\backslash A\backslash^2) \geq 0$*for* $1 \leq \ \leq n-1$. *When* $2 \leq \ \leq n-2$, *assume further that* M^n *has flat normal bundle, then* $H^{l,p}(L^p(M)) = \{0\}$, *where* δ *is a constant satisfying the following inequality:*

$$\delta > \frac{l(n-l)p^2}{4n[p-1+(p-1)^2k_p]}$$

Remark: Our theorem extends Han's conclusion.

2. Preliminary

Let $\iota: M \to N$ be an n-dimensional submanifold isometrically immersed in an $(n+m)$-dimensional Riemannian manifold $(N, \bar{g})$. Fix a point $x \in M$ and a local orthonormal frame $\{e_1, \ldots, e_{n+m}\}$ of N such that $\{e_1, \ldots, e_n\}$ are tangent fields of M at x and $\{e_n, \ldots, e_{n+m}\}$ is a local orthonormal frame of normal bundle NM. In this paper, we also adopt the following index ranges: $1 \leq i, j \ldots \leq n$, $n + 1 \leq \alpha, \beta, \ldots \leq n + m$. $A: \Gamma(TM) \times \Gamma(TM) \to \Gamma(NM)$ is the second fundamental form. Then

$$A(X,Y) = \sum_\alpha \langle \overline{\nabla}_X Y, e_\alpha \rangle e_\alpha$$

We denote $h_{ij}^\alpha = \langle \overline{\nabla}_{e_i} e_j, e_\alpha \rangle$. Then, provide the square norm $\backslash\backslash A \backslash\backslash^2$ of the second fundamental form and the mean curvature vector H:

$$\setminus A \setminus^2 = \sum_\alpha \sum_{i,j} (h_{ij}^\alpha)^2, H = \frac{1}{n} H^\alpha e_\alpha = \frac{1}{n} \sum_\alpha \sum_i h_{ii}^\alpha e_\alpha$$

A submanifold M is said to be minimal if $H = 0$ identically. The traceless second fundamental form ϕ is defined by

$$\phi(X,Y) = A(X,Y) - \langle X,Y \rangle H,$$

for any vector fields $X, Y \in \Gamma(TM)$. It is easy to check that

$$\setminus \phi \setminus^2 = \setminus A \setminus^2 - n \setminus H \setminus^2.$$

Definition 2.1 (Definition 4.). *M^n is a minimal submanifold on the sphere S^{n+m}. If M satisfies the following inequality*

$$\int_M (|\nabla\varphi|^2 - \delta|A|^2)dv \geq 0$$

when φ is a smooth compact support function on M (i.e. $\varphi \in C_0^\infty(M)$), then M is said to be δ-super stable.

It is easy to see that M is δ-super stable in S^{n+m}, if and only if $(\Delta + \delta|A|^2) = 0$, this is also equivalent to $\lambda_1(\Delta + \delta|A|^2) \geq 0$ [10].

Next, we will give some important lemmas.

Lemma 2.2 [15] *For $p \geq 2$ and ≥ 1, let be a p-harmonic ℓ-form on an n-dimensional complete Riemannian manifold M. Then we have*

$$|\nabla(|\omega|^{p-2}\omega)|^2 \geq (1 + k_p)|\nabla|\omega|^{p-1}|^2,$$

where

$$k_p = \begin{cases} \dfrac{1}{max\{l, n-l\}}, & if\ p = 2 \\ \dfrac{1}{(p-1)^2} min\{1, \dfrac{(p-1)^2}{n-1}\}, & if p > 2\ and\ l = 1 \\ 0, & if p > 2\ and\ 1 < l \leq n-1 \end{cases}$$

Lemma 2.3 (Lemma 2.5 [16]) *For any closed l-form $\omega \in \Omega^l(M)$ and $f \in C^\infty(M)$, we have* $|d(f\omega)| = |df \wedge \omega| \leq |df| \cdot |\omega|$.

3. The proof of Theorem 1.1

Proof. Let be any p-harmonic ℓ-form on M with $1 \leq l \leq n-2$, using the proof of Theorem 3.1 [10], we have

(1) $$\frac{1}{2}\Delta|\omega|^2 \geq |\nabla\omega|^2 + \langle\Delta\omega, \omega\rangle + l(n-l)|\omega|^2 - \frac{l(n-l)}{n}|A|^2|\omega|^2$$

Applying (1) to the form $|\omega|^{p-2}\omega$, we obtain

(2) $$\frac{1}{2}\Delta|\omega|^{2(p-1)} \geq |\nabla(|\omega|^{p-2}\omega)|^2 - \langle(d\delta + \delta d)|\omega|^{p-2}\omega, |\omega|^{p-2}\omega\rangle + l(n-l)|\omega|^2 - \frac{l(n-l)}{n}|A|^2|\omega|^2$$

By direct calculation, we have

(3) $$\frac{1}{2}\Delta|\omega|^{2(p-1)} \geq |\omega|^{p-1}\Delta|\omega|^{p-1} + |\nabla|\omega|^{p-1}|^2$$

Combining (2), (3), and $\delta(|\omega|^{p-2}\omega) = 0$, it follows that

$$\begin{aligned} |\omega|^{p-1}\Delta|\omega|^{p-1} \geq & (|\nabla(|\omega|^{p-2}\omega)|^2 - |\nabla|\omega|^{p-1}|^2) \\ & - \langle(d\delta + \delta d)|\omega|^{p-2}\omega, |\omega|^{p-2}\omega\rangle \\ & - \frac{l(n-l)}{n}|A|^2|\omega|^{2(p-1)} + l(n-l)|\omega|^{2(p-1)} \end{aligned}$$

Applying Lemma 2.2 to the above inequality and dividing both sides of the inequality by $\backslash \omega \backslash^{p-2}$, we have

$$(4)\quad \backslash \omega \backslash \Delta \backslash \omega \backslash^{p-1} \geq (p-1)^2 k_p \backslash \omega \backslash^{p-2} \backslash \nabla \backslash \omega \backslash\backslash^2) - \langle \delta d(\backslash \omega \backslash^{p-2} \omega), \omega \rangle - \frac{l(n-l)}{n} \backslash A \backslash^2 \backslash \omega \backslash^p + l(n-l) \backslash \omega \backslash^p$$

Let r(x) be the geodesic distance on M from a fixed point $x_0 \in M$ to x. Then we take a cutoff function $\phi \in C_0^\infty(M)$ satisfying

$$(5)\quad \varphi(x) = \begin{cases} 1 \quad, on\ B_r(x_0), \\ \in [0,1]\ and\ \backslash \nabla\varphi \backslash \leq \frac{2}{r}, on\ B_{2r}(x_0) \backslash B_r(x_0), \\ 0, on\ M \backslash B_{2r}(x_0), \end{cases}$$

where $B_r(x_0)$ is an open geodesic ball on *M* with r as the radius and x_0 as the center.
Multiplying both sides of the inequality (3) by φ^2 and integrating over *M*, it holds

$$(6)\quad \begin{aligned} &\int_M \varphi^2 \backslash \omega \backslash \Delta \backslash \omega \backslash^{p-1} dv \\ &\geq (p-1)^2 k_p \int_M \varphi^2 \backslash \omega \backslash^{p-2} \backslash \nabla \backslash \omega \backslash\backslash^2 dv - \int_M \langle d \backslash \omega \backslash^{p-2} \omega, d(\varphi^2 \omega) \rangle\, dv \\ &\quad + l(n-l) \int_M \varphi^2 \backslash \omega \backslash^p\, dv - \frac{l(n-l)}{n} \int_M \varphi^2 \backslash A \backslash^2 \backslash \omega \backslash^p dv \end{aligned}$$

For the left part of the inequality, using integration by parts yields

$$(7)\quad \begin{aligned} &\int_M \varphi^2 \backslash \omega \backslash \Delta \backslash \omega \backslash^{p-1} dv = -\int_M \langle \nabla(\varphi^2 \backslash \omega \backslash), \nabla \backslash \omega \backslash^{p-1} \rangle\, dv \\ &= -2(p-1) \int_M \varphi \backslash \omega \backslash^{p-1} \langle \nabla\varphi, \nabla \backslash \omega \backslash \rangle\, dv - (p-1) \int_M \varphi^2 \backslash \omega \backslash^{p-2} \backslash \nabla \backslash \omega \backslash\backslash^2\, dv \\ &\leq 2(p-1) \int_M \varphi \backslash \omega \backslash^{p-1} \backslash \nabla\varphi \backslash\backslash \nabla \backslash \omega \backslash\backslash dv - (p-1) \int_M \varphi^2 \backslash \omega \backslash^{p-2} \backslash \nabla \backslash \omega \backslash\backslash^2\, dv \end{aligned}$$

By using Lemma 2.3 and $d\omega = 0$ for the second part to the right of the inequality, we get

$$(8)\quad \begin{aligned} &\int_M \langle d \backslash \omega \backslash^{p-2} \omega, d(\varphi^2 \omega) \rangle\, dv \leq \int_M \backslash d \backslash \omega \backslash^{p-2} \wedge \omega \backslash \cdot \backslash d\varphi^2 \wedge \omega \backslash\, dv \\ &= 2(p-2) \int_M \varphi \backslash \omega \backslash^{p-1} \backslash \nabla\varphi \backslash\backslash \nabla \backslash \omega \backslash\backslash dv \end{aligned}$$

Combining (6), (7), and (8), it implies that

$$(9)\quad \begin{aligned} &0 \geq -2(2p-3) \int_M \varphi \backslash \omega \backslash^{p-1} \backslash \nabla\varphi \backslash\backslash \nabla \backslash \omega \backslash\backslash\, dv \\ &\quad + [p-1 + (p-1)^2 k_p \int_M \varphi^2 \backslash \omega \backslash^{p-2} \backslash \nabla \backslash \omega \backslash\backslash^2\, dv \\ &\quad + l(n-l) \int_M \varphi^2 \backslash \omega \backslash^p\, dv - \frac{l(n-l)}{n} \int_M \varphi^2 \backslash A \backslash^2 \backslash \omega \backslash^p dv \end{aligned}$$

Next, we shall give some estimates of (9). Firstly, by Schwarz's inequality, we have

$$2\int_M \varphi|\omega|^{p-1}|\nabla\varphi||\nabla|\omega||\,dv$$

(10) $$\leq \varepsilon_1\int_M \varphi^2|\omega|^{p-2}|\nabla|\omega||^2\,dv + \frac{1}{\varepsilon_1}\int_M |\nabla\varphi|^2|\omega|^p\,dv$$

where $\varepsilon_1 > 0$ is a constant. Secondly, due to $\lambda_l(+ \delta|A|^2) \geq 0$, we have

$$\delta\int_M \varphi^2|A|^2|\omega|^p dv \leq \int_M |\nabla(\varphi|\omega|^{\frac{p}{2}})|^2\,dv$$

(11) $$\leq \frac{p^2}{4}(1+\varepsilon_2)\int_M \varphi^2|\omega|^{p-2}|\nabla|\omega||^2\,dv + (1+\frac{1}{\varepsilon_2})\int_M |\nabla\varphi|^2|\omega|^p\,dv$$

where $\varepsilon_2 > 0$ is a constant. Therefore, combining (9), (10), and (11), it follows that

$$\mathrm{B}\int_M \varphi^2|\omega|^{p-2}|\nabla|\omega||^2\,dv + l(n-l)\int_M \varphi^2|\omega|^p\,dv$$

(12) $$\leq C\int_M |\nabla\varphi|^2|\omega|^p\,dv,$$

where

$$B = p - 1 + (p-1)^2k_p - (2p-3)\varepsilon_1 - \frac{l(n-l)p^2}{4n\delta}(1+\varepsilon_2),$$
$$C = \frac{2p-3}{\varepsilon_1} + \frac{l(n-l)}{n\delta}(1+\frac{1}{\varepsilon_2})$$

Then we can choose sufficiently small ε_1 and ε_2 such that $\mathrm{B} > 0$. Since φ is a compactly supported nonnegative smooth function on M and $|\nabla\varphi|^2 \leq \frac{4}{r^2}$, we have

(13) $$\mathrm{B}\int_{B_r(x_0)} |\omega|^{p-2}|\nabla|\omega||^2\,dv + l(n-l)\int_{B_r(x_0)} |\omega|^p\,dv$$

$$\leq \mathrm{B}\int_M \varphi^2|\omega|^{p-2}|\nabla|\omega||^2\,dv + l(n-l)\int_M \varphi^2|\omega|^p\,dv$$

$$\leq \frac{4C}{r^2}\int_{B_{2r}(x_0)} |\omega|^p\,dv$$

Since $\omega \in L^P(M)$, letting $r \to +\infty$ in (13) , it follows that
$|\omega|^p|\nabla|\omega||^2 = 0$, $|\omega|^p = 0$
Then $H^{l,p}(L^p(M)) = \{0\}$, $1 \leq l \leq n-2$. Due to Poincaré duality, we have $H^{l,p}(L^p(M)) = \{0\}$, $1 \leq l \leq n-1$.

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