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Dynamics of a Reaction-Diffusion System with Quiescence

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Abstract: In this paper, the dynamical behavior of a reaction-diffusion system with quiescence in a closed environment is investigated. The global existence of the solution is obtained by the upper and lower solution method, and the dissipative structure of the system is derived by constructing Lyapunov functions.

Keywords: Reaction-diffusion; Dissipative; Quiescence

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1. Introduction

Reaction-diffusion system can be coupled for two different dynamics acting on the same space in several ways. If one of the two vector fields disappears, then a given dynamics is coupled to a quiescent phase. Quiescent phases appear in different ways in population models and under different names such as quiescent state [1], dormancy [2], resting phase [3], and ecological refuge. Generally, these phases may have drastic effects on the dynamics.

Introducing quiescent phases will suppress oscillations and even make them disappear [4]. Hadeler and Lewis [5] presented and discussed briefly the following model:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = D\Delta u(x,t) + f(u(x,t)) - \gamma_1 u(x,t) + \gamma_2 v(x,t) \\ \dfrac{\partial v(x,t)}{\partial t} = \gamma_1 u(x,t) - \gamma_2 v(x,t) \end{cases} \tag{1}$$

The model (1) describes a population in which the individuals alternate between mobile and non-mobile states, and only the mobile reproduces, where f is the reproduction function, $\gamma_1(\gamma_2)$ stands for the conversion rate from mobile(non-mobile) to non-mobile(mobile). For invertebrates living in small ponds in arid climates, such behavior is typical, which dry up and reappear under the influence of rainfall. However, Hadeler and Lewis did not show further mathematical analysis. We will supplement some mathematical results in this paper.

Inspired by the study [5], the purpose of this paper is to consider the system (1) with the non-negative initial conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u + f(u) - \gamma_1 u + \gamma_2 v, & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = \gamma_1 u - \gamma_2 v, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{2}$$

where Ω is a bounded domain in $R^N (N \geq 1)$ with a smooth boundary $\partial\Omega$; is the unit outer normal, and no flux boundary condition is imposed, which explains that the system is closed. f denotes the general growth non-linearity with carrying capacity. In practical problems, suppose that the growth function $f \in C^1(R_+, R)$ satisfies the following condition:

$$f(0) = f(K) = 0, \tag{3}$$

and there exists $K > 0$ such that $f(u) < 0$ for all $u > K$,

where K is the carrying capacity of species.

2. Global existence of solutions

The existence, boundedness, and uniqueness of globally defined solutions for the reaction-diffusion system (2) are shown in this section.

Firstly, we consider the associated ordinary differential equation (ODE) system:

$$\begin{cases} \dfrac{du}{dt} = f(u) - \gamma_1 u + \gamma_2 v, \\ \dfrac{dv}{dt} = \gamma_1 u - \gamma_2 v \end{cases} \tag{4}$$

for $t > 0$. The positive quadrant of the phase plane $\{(u, v) : u \geq 0, v \geq 0\}$ is invariant, which is easy to verify. The following Lemma guarantees that the system (4) is bounded.

Lemma 2.1. *Suppose that f satisfies (3), then any solution of (4) with positive initial value is positive and bounded.*

Proof. Since $\{(u, v) : u = 0\}$ and $\{(u, v) : v = 0\}$ are invariant manifolds of (4), then the first quadrant is an invariant region for (4). Therefore, the solutions of (4) with the initial values $u(0)>0$ and $v(0)>0$ are positive.

In addition, the two equations of (4) and the condition (3) also show that $[0, \bar{K}] \times [0, \gamma_1 \bar{K} / \gamma_2]$ is a positive invariant rectangle for (4) for any $\bar{K} \geq K$. Therefore, the solution of (4) is bounded.

Now, we demonstrate the global existence of solutions for (2).

Theorem 2.2. *Assume that* $\gamma_1, \gamma_2 > 0$, *f satisfies (3) and* $\Omega \subset R^n$ *is a bounded domain with smooth boundary.*

(a) If $u_0(x)\ge 0,\ v_0(x)\ge 0,$ then (2) has a unique solution $(u(x,t),v(x,t))$ such that $u(x,t)>0,\ v(x,t)>0$ for $t\in(0,\infty)$ and $x\in\overline{\Omega}$;

(b) $u(x,t)\le C_1,\ v(x,t)\le C_2$, where C_1 and C_2 are constants only depending on γ_1,γ_2 K and the initial value $u_0(x), v_0(x)$.

Proof. The existence of a unique local solution of (2) can be obtained by referring to Theorem 14.2 [6] or Theorem 3.3.3 and Exercise 3 [7].

Define $M(u,v)=f(u)-\gamma_1 u+\gamma_2 v,\quad N(u,v)=\gamma_1 u-\gamma_2 v,$

then $M_v\ge 0$ and $N_u\ge 0$ in $\overline{R_+^2}=\{u\ge 0, v\ge 0\}$, furthermore, (2) is a quasi-monotone system [8,9].

Let $(\underline{u}(x,t),\underline{v}(x,t))=(0,0)$ and $(\overline{u}(x,t),\overline{v}(x,t))=(u^*(t),v^*(t))$, where $(u^*(t),v^*(t))$ is the unique solution to (4) with the initial value $u^*(0)=u^*, v^*(0)=v^*,$ where $u^*=\sup_{\overline{\Omega}}u(x)$ and $v^*=\sup_{\overline{\Omega}}v(x)$, it shows $0\le u(x)\le u^*$ and $0\le v(x)\le v^*$. Consider the following problem:

$$\begin{cases}\dfrac{\partial\overline{u}(x,t)}{\partial t}-D\Delta\overline{u}(x,t)-M(\overline{u}(x,t),\overline{v}(x,t))=0\ge 0, & x\in\Omega,\ t>0,\\ \dfrac{\partial\overline{v}(x,t)}{\partial t}-N(\overline{u}(x,t),\overline{v}(x,t))=0\ge 0, & x\in\Omega,\ t>0,\\ \dfrac{\partial\overline{u}(x,t)}{\partial n}=0, & x\in\partial\Omega,\ t>0,\\ \overline{u}(x,0)=u^*,\overline{v}(x,0)=v^*, & x\in\Omega,\end{cases}$$

and

$$\begin{cases}\dfrac{\partial\underline{u}(x,t)}{\partial t}-D\Delta\underline{u}(x,t)-M(\underline{u}(x,t),\underline{v}(x,t))=0\le 0, & x\in\Omega,\ t>0,\\ \dfrac{\partial\underline{v}(x,t)}{\partial t}-N(\underline{u}(x,t),\underline{v}(x,t))=0\le 0, & x\in\Omega,\ t>0,\\ \dfrac{\partial\underline{u}(x,t)}{\partial n}=0, & x\in\partial\Omega,\ t>0,\\ \underline{u}(x,0)=0,\underline{v}(x,0)=0, & x\in\Omega.\end{cases}$$

Thus, it follows from the definition of lower/upper-solution that $(\underline{u}(x,t),\underline{v}(x,t))=(0,0)$ and $(\overline{u}(x,t),\overline{v}(x,t))=(u^*(t),v^*(t))$ are the lower-solution and upper-solution to (2) respectively, which can refer to Definition 8.1.2 [8] or Definition 5.2.1 [9]. Therefore, it can be concluded that (2) has a unique globally defined solution $(u(x,t),v(x,t))$ which satisfies $0\le u(x,t)\le u^*(t),\quad 0\le v(x,t)\le v^*(t),\quad t\ge 0$ from Theorem 8.3.3 [8] or Corollary 5.2.11 [9]. The strong maximum principle implies that $u(x,t),v(x,t)>0$ when $t>0$ for all $x\in\overline{\Omega}$. Moreover, from Lemma 2.1., we can see that $u(x,t)\le u^*(t)\le\max\{u^*,K\}:=C_1,$ and $v(x,t)\le v^*(t)\le\max\{v^*,\gamma_1K/\gamma_2\}:=C_2.$

The boundedness of $(u(x,t),v(x,t))$ implies the global existence of the solutions, which completes the proof of parts (a) and (b).

3. Dissipative structure

Now, we can further acquire the dissipative property of the reaction-diffusion system (2).

Theorem 3.1. *The system (2) is dissipative and there is no periodic solution.*

Proof. Let

$$W(u,v)=\frac{\gamma_1 D}{2}\int_\Omega|\nabla u|^2dx-\gamma_1\int_\Omega F(x,u)dx+\frac{1}{2}\int_\Omega(\gamma_1 u-\gamma_1 v)^2dx, \tag{5}$$

where $F(x,u)=\int_0^u f(x,s)ds$. Then

$$\begin{aligned}\dot{W}&=\gamma_1 D\int_\Omega\nabla u\cdot\nabla u_t dx-\gamma_1\int_\Omega f(u)u_t dx+\int_\Omega(\gamma_1 u-\gamma_2 v)(\gamma_1 u_t-\gamma_2 v_t)dx\\&=-\gamma_1 D\int_\Omega\Delta u u_t dx-\gamma_1\int_\Omega f(u)u_t dx+\gamma_1\int_\Omega(\gamma_1 u-\gamma_2 v)u_t dx-\gamma_2\int_\Omega(\gamma_1 u-\gamma_2 v)v_t dx\\&=-\gamma_1\int_\Omega(D\Delta u+f(u)-\gamma_1 u+\gamma_2 v)u_t dx-\gamma_2\int_\Omega(\gamma_1 u-\gamma_2 v)v_t dx\\&=-\gamma_1\int_\Omega u_t^2-\gamma_2 v_t^2 dx\le 0,\end{aligned} \tag{6}$$

which gives us a hint of what we might expect.

Notice that if u is a steady state of $u_t=d\Delta u+f(u)$ if and only if $(u,\gamma_1 u/\gamma_2)$ is a steady state of (2). Regarding the steady states of (2), we obtain the following results:

Corollary 3.2.

(i) If $\dot{W}=0$, then $u_t\equiv 0$ and $v_t\equiv 0$, which implies that (u, v) is a steady state of (2).

(ii) If u is the unique steady state solution of $u_t=d\Delta u+f(u)$, then $(u,\gamma_1 u/\gamma_2)$ is globally asymptotically stable for system (2).

Remark 3.3. *For a class of systems that simulate the random dispersal of the pollutant, but ignore the small mobility of the infected population. By applying the phase plane method to the endpoints of one-dimensional intervals under homogeneous Dirichlet boundary conditions, the steady-state bifurcation pattern is analyzed in detail, while our approach is quite different and more representative.*

Disclosure statement

The author declares no conflict of interest.

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