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Studying the Two New Convolutions of Fractional Fourier Transform by Using Dirac Notation

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Abstract: Based on quantum mechanical representation and operator theory, this paper restates the two new convolutions of fractional Fourier transform (FrFT) by making full use of the conversion relationship between two mutual conjugates: coordinate representation and momentum representation. This paper gives full play to the efficiency of Dirac notation and proves the convolutions of fractional Fourier transform from the perspective of quantum optics, a field that has been developing rapidly. These two new convolution methods have potential value in signal processing.

Keywords: Fractional Fourier transform; Convolution theorem; Quantum mechanical representation; Dirac notation

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1. Introduction

In 1980, Namis first proposed the fractional Fourier transform to solve equations in quantum mechanics, which did not attract much attention at that time [1]. In 1987, Mendlvic and Ozaktas described the propagation of light field in a second-order gradient medium using fractional Fourier transform [2]. In October 1993, optical expert Lohmann AW used the properties of Fourier transform to combine three concepts (image rotation, Wigner rotation, and fractional Fourier transform), in order to explain the physical significance of fractional Fourier transform [3,4]. Fractional Fourier transform has been extensively studied in the field of optics up to this day. In 1999, Chountasis discussed about fractional Fourier transform based on quantum optical representation and operator theory. It was proposed that the integral kernel of fractional Fourier transform could be expressed as the matrix element of the operator between coordinate representation and momentum representation. In 2003, fractional Fourier transform was extended to the entangled state representation based on quantum optical representation and operator theory, and a complex fractional Fourier transform was proposed; a quantum optical representation of the fractional Fourier transform and the complex fractional Fourier transform was put forward [5,6]. In 2011, fractional Fourier transform was extended to the three-mode entanglement representation, and a three-mode fractional Fourier transform was obtained [7,8]. Since fractional Fourier transform can overcome the limitations of the classical Fourier transform signal analysis, it can be compared and analyzed in multiple fractional domains, so as to obtain the “global” meaning of the optimal processing results. In view of that, the convolution of fractional Fourier transform has been widely concerned in recent years [9-11]. The convolution theory of fractional Fourier transform has become one of the most active fields of research in regard to signal processing and is widely used in optics, radar, communication, as well as image and signal processing [12-16].

This paper mainly studies the two new convolution theorems of fractional Fourier transform from the perspective of quantum mechanics. In the second section, the quantum optical representation of fractional

Fourier transform, the completeness of representation, and other properties are briefly reviewed. In the third section, the two new convolution theorems of fractional Fourier transform [17,18] proposed by Anh PK and several other researchers are restated by using Dirac notation, and the two new convolution theorems are proven by the relationship between momentum representation and coordinate representation. A summary is given in the last section of this paper.

2. Convolution theorem for fractional Fourier transform in the context of quantum mechanics

Fractional Fourier transform is converted into a form represented by the representation theory in quantum mechanics. The optical fractional Fourier transform (order) in one-dimensional cases is as follows:

$$F_\alpha\left[g(x)\right]=\sqrt{\frac{\exp\left[-i\left(\frac{\pi}{2}-\alpha\right)\right]}{2\pi\sin\alpha}}\times\int\exp\left[\frac{i}{2}\left(\frac{p^2+x^2}{\tan\alpha}-\frac{2px}{\sin\alpha}\right)\right]g(x)dx\equiv G(p) \tag{1}$$

where "e" to the exponent is the integral kernel. The eigenstates of coordinate and momentum in Fork space are expressed as follows:

$$|x\rangle=\frac{1}{\pi^{1/4}}\exp\left[-\frac{x^2}{2}+\sqrt{2}xa^+-\frac{a^{+2}}{2}\right]|0\rangle \tag{2}$$

$$|p\rangle=\pi^{-1/4}\exp\left(-\frac{p^2}{2}+\sqrt{2}ipa^+ +\frac{a^{+2}}{2}\right)|0\rangle \tag{3}$$

With integration within an ordered product of operators (IWOP technique), as well as equation (2) and equation (3), the following can be obtained:

$$|x\rangle\langle p|=\pi^{-1/2}:\exp\left\{-\frac{x^2}{2}+\sqrt{2}xa^+-\frac{a^{+2}}{2}-\frac{p^2}{2}-i\sqrt{2}pa+\frac{a^2}{2}-a^+a\right\}: \tag{4}$$

By multiplying $\int_{-\infty}^{\infty}dp|p\rangle$ and $\int_{-\infty}^{\infty}dx\langle x|$ on both sides of the integral kernel, and using equation (4) as well as the IWOP technique, the following can be obtained:

$$\iint dxdp|p\rangle\exp\left(\frac{ix^2+ip^2}{2\tan\alpha}-\frac{ixp}{\sin\alpha}\right)\langle x|$$

$$=\left(\sqrt{\pi}\right)^{-1}:\iint dxdp\exp\left(\begin{array}{l}-\frac{p^2}{2}+i\sqrt{2}pa^+ +\frac{a^{+2}}{2}+\frac{ix^2}{2\tan\alpha}+\frac{ip^2}{2\tan\alpha}\\-\frac{ixp}{\sin\alpha}-\frac{x^2}{2}+\sqrt{2}xa-\frac{a^2}{2}-a^+a\end{array}\right):$$

$$=\sqrt{\frac{2}{1-i\cot\alpha}}:\iint dxdp\exp\left\{\begin{array}{l}-p^2+i\sqrt{2}p\left[a^+-\frac{a}{\sin\alpha\left(1-i\cot\alpha\right)}\right]\\+\frac{a^{+2}}{2}-\frac{a^2}{2}\left[1-\frac{1}{\left(1-i\cot\alpha\right)}\right]-a^+a\end{array}\right\}:$$

$$=\sqrt{\frac{2}{1-i\cot\alpha}}:\exp\left[\left(ie^{-i\alpha}-1\right)a^+a\right]:$$

$$=\sqrt{2\pi i\sin\alpha e^{-i\alpha}}\exp\left[i\left(\frac{\pi}{2}-\alpha\right)a^+a\right] \tag{5}$$

By virtue of the completeness between the coordinate eigenvector and momentum eigenvector [19,20],

$$\int_{-\infty}^{\infty}dx\left|x\right\rangle\left\langle x\right|=1,\quad \int_{-\infty}^{\infty}dp\left|p\right\rangle\left\langle p\right|=1 \tag{6}$$

the following can be obtained:

$$\exp\left\{\frac{i\left(x^2+p^2\right)}{2\tan\alpha}-\frac{ixp}{\sin\alpha}\right\}=\sqrt{2\pi i\sin\alpha e^{-i\alpha}}\left\langle p\right|\exp\left[i\left(\frac{\pi}{2}-\alpha\right)a^+a\right]\left|x\right\rangle \tag{7}$$

When $\alpha=\pi/2$, equation (9) becomes the Fourier transform in the normal sense. Therefore, if $g\left(x\right)$ is regarded as the wave function of $\left\langle x|g\right\rangle$, the quantum state $\left|g\right\rangle$ is the wave function in the coordinate representation; then, its fractional Fourier transform becomes as such:

$$\begin{aligned}&F_\alpha\left(g\left(x\right)\right)\\&=\int_{-\infty}^{\infty}dx\left\langle p\right|\exp\left[i\left(\frac{\pi}{2}-\alpha\right)a^+a\right]\left|x\right\rangle\left\langle x|g\right\rangle\\&=\left\langle p\right|\exp\left[i\left(\frac{\pi}{2}-\alpha\right)a^+a\right]\left|g\right\rangle\\&\equiv G\left(p\right)\end{aligned} \tag{8}$$

which implies

$$\left|G\right\rangle=F\left|g\right\rangle \tag{9}$$

where

$$F=\exp\left[i\left(\frac{\pi}{2}-\alpha\right)a^+a\right] \tag{10}$$

$|G\rangle$ is the fractional transformed state of $|g\rangle$. In a study [21], a new prove of quantum mechanical representation transformation for the convolution theorem under FrFT was put forward, which seemed simple and elegant. The essence of FrFT can be seen more clearly from the perspective of representation transformation in quantum mechanics. In the context of quantum mechanics, the FrFT of some wave functions can be deduced more directly and succinctly. The definition of convolution theorem for FrFT is as follows (FrFT of $h(x)$ is $H_\alpha(p)$):

$$F_\alpha\left(h(x)\right) = \Im_\alpha(p)\wp_\alpha(p)e^{-ik(\alpha)p^2} \equiv H_\alpha(p) \tag{11}$$

where

$$(f \bullet g)(x) = \frac{c(\alpha)}{\sqrt{2\pi}} e^{-ik(\alpha)x^2}\left(\tilde{f} * \tilde{g}\right)(x) \equiv h(x) \tag{12}$$

$$\left(\tilde{f} * \tilde{g}\right)(x) = \int_{-\infty}^{\infty} \tilde{f}(y)\tilde{g}(x-y)dy \tag{13}$$

$$\tilde{f}(x) \equiv f(x)e^{ik(\alpha)x^2}, \quad k(\alpha) = \frac{1}{2}(\cot\alpha), \quad c(\alpha) = \sqrt{1 - i\cot\alpha} \tag{14}$$

With equation (8), the new prove of quantum mechanical representation transformation for the convolution theorem under FrFT is as follows:

$$F_\alpha\left(h(x)\right) = \int_{-\infty}^{\infty} dx\langle p|F|x\rangle\langle x|h\rangle = \langle p|F|h\rangle = \langle p|H\rangle \equiv H_\alpha(p) \tag{15}$$

Expressing the function $h(x)$ in coordinate representation,

$$\begin{gathered}\langle x|h\rangle = \langle x|f \bullet g\rangle \\ \left(\tilde{f} * \tilde{g}\right)(x) = \langle x|\tilde{f} * \tilde{g}\rangle\end{gathered} \tag{16}$$

equation (15) can then be rewritten as follows:

$$\begin{aligned}
H_\alpha(p) &= \int_{-\infty}^{\infty} dx\langle p|F|x\rangle\langle x|f \bullet g\rangle \\
&= \frac{c(\alpha)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\langle p|F|x\rangle\langle x|\tilde{f} * \tilde{g}\rangle e^{-ik(\alpha)x^2} \\
&= \frac{c(\alpha)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dxdy\langle p|F|x+y\rangle\langle x|g\rangle\langle y|f\rangle e^{-ik(\alpha)2yx} \\
&= \langle p|F|f\rangle\langle p|F|g\rangle e^{-ik(\alpha)p^2} \\
&= F_\alpha\left(f(x)\right)F_\alpha\left(g(x)\right)e^{-ia(\alpha)p^2} \\
&= \langle p|F\rangle\langle p|G\rangle e^{-ia(\alpha)p^2} \\
&= \Im_\alpha(p)\wp_\alpha(p)e^{-ia(\alpha)p^2}
\end{aligned} \tag{17}$$

3. Two new convolutions represented by quantum mechanical representation

In this section, two new convolutions associated with FRFT in the context of quantum mechanics are introduced.

Definition 1: We define the convolution operation $\Box$ by

$$h(s) := (f \Box g)(s) = \frac{c}{\sqrt{2\pi}} e^{ia\left(2u^2 - 2su + \frac{s}{ab} - \frac{u}{ab}\right)} \int_{-\infty}^{\infty} f(u) \times g\left(s - u + \frac{1}{2ab}\right) du \tag{18}$$

Definition 2: We define the product $f \otimes g$ by

$$h(s) := (f \otimes g)(s) = \frac{c}{\sqrt{2\pi}} e^{ia\left(2u^2 - 2su - \frac{s}{ab} + \frac{u}{ab}\right)} \int_{-\infty}^{\infty} f(u) \times g\left(s - u - \frac{1}{2ab}\right) du \tag{19}$$

The FrFT of $h(s)$ is $H_\alpha(p)$. If $\Im_\alpha[f](x)$ and $\Im_\alpha[g](x)$ are the fractional Fourier transforms of $g(x)$ and $f(x)$, respectively, we can prove the following theorem:

$$\Im_\alpha[f \Box g](x) = \psi(x) \Im_\alpha[f](x) \Im_\alpha[g](x) \tag{20}$$

Proving Definition 1, equation (18) can be written as such:

$$h(s) := (f \Box g)(s) = \frac{c}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du \langle u | f \rangle \left\langle s - u + \frac{1}{2ab} \middle| g \right\rangle e^{ia\left(2u^2 - 2su + \frac{s}{ab} - \frac{u}{ab}\right)} \tag{21}$$

The general fractional Fourier transform is as follows:

$$H_\alpha(p) = \frac{c}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \langle p | F | s \rangle \int_{-\infty}^{\infty} du \langle u | f \rangle \left\langle s - u + \frac{1}{2ab} \middle| g \right\rangle \exp\left[ia\left(2u^2 - 2su + \frac{s}{ab} - \frac{u}{ab}\right)\right] \tag{22}$$

The FrFT of $h(s)$ is $H_\alpha(p)$, where

$$
\begin{aligned}
&H_\alpha(p)\\
&=\frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ds\langle p|F|s\rangle\int_{-\infty}^{\infty} du\langle u|f\rangle\left\langle s-u+\frac{1}{2ab}\middle|g\right\rangle \exp\left[ia\left(2u^2-2su+\frac{s}{ab}-\frac{u}{ab}\right)\right]\\
&=\frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ds\left\langle p|F\middle|s+u-\frac{1}{2ab}\right\rangle\int_{-\infty}^{\infty} du\langle u|f\rangle\langle s|g\rangle\\
&\exp\left\{ia\left[\left(2u^2-2u\left(s+u-\frac{1}{2ab}\right)+\left(\frac{s+u-\frac{1}{2ab}}{ab}\right)-\frac{u}{ab}\right)\right]\right\}\\
&=\frac{c}{\sqrt{2\pi}}\iint dsdu\left\langle p|F\middle|s+u-\frac{1}{2ab}\right\rangle\langle u|f\rangle\langle s|g\rangle\\
&\exp\left\{ia\left[\left(2u^2-2u\left(s+u-\frac{1}{2ab}\right)+\left(\frac{s+u-\frac{1}{2ab}}{ab}\right)-\frac{u}{ab}\right)\right]\right\}\\
&=\frac{c}{\sqrt{2\pi}}\iint dsdu\left\langle p|F\middle|s+u-\frac{1}{2ab}\right\rangle\langle u|f\rangle\langle s|g\rangle\\
&\exp\left\{-ia\left[2su-\frac{u}{ab}-\frac{s}{ab}+\frac{1}{2a^2b^2}\right]\right\}
\end{aligned}
\tag{23}
$$

Rewriting $\left\langle p|F\middle|s+u-\frac{1}{2ab}\right\rangle$ explicitly according to equation (11), the following can be obtained:

$$
\begin{aligned}
&\left\langle p|F\middle|s+u-\frac{1}{2ab}\right\rangle \exp\left\{-ia\left[2su-\frac{u}{ab}-\frac{s}{ab}+\frac{1}{2a^2b^2}\right]\right\}\frac{c}{\sqrt{2\pi}}\\
&=\left(\frac{c}{\sqrt{2\pi}}\right)^2\exp\left\{\begin{array}{l}\dfrac{i\left[\left(s+u-\frac{1}{2ab}\right)^2+p^2\right]}{2\tan\alpha}-\dfrac{i\left(s+u-\frac{1}{2ab}\right)p}{\sin\alpha}\\ -ia\left(2su-\dfrac{u}{ab}-\dfrac{s}{ab}+\dfrac{1}{2a^2b^2}\right)\end{array}\right\}
\end{aligned}
\tag{24}
$$

Substituting $a=\frac{1}{2\tan\alpha}$, $b=\frac{1}{\cos\alpha}$, $ab=\frac{1}{2\tan\alpha\cos\alpha}$, $ab^2=\frac{1}{2\cos^2\alpha\tan\alpha}$, and $\frac{1}{2ab}=\tan\alpha\cos\alpha$ into equation (24), the following can be obtained:

$$\left(\frac{c}{\sqrt{2\pi}}\right)^2\exp\left\{\begin{array}{l}\frac{i\left[\left(s+u-\frac{1}{2ab}\right)^2+p^2\right]}{2\tan\alpha}-\frac{i\left(s+u-\frac{1}{2ab}\right)p}{\sin\alpha}\\ +\frac{i}{b}s+\frac{i}{b}u-\frac{i}{2ab^2}-2iasu\end{array}\right\}$$

$$=\left(\frac{c}{\sqrt{2\pi}}\right)^2\exp\left\{\begin{array}{l}\frac{i\left[\left(s+u-\tan\alpha\cos\alpha\right)^2+p^2\right]}{2\tan\alpha}-\frac{i\left(s+u-\tan\alpha\cos\alpha\right)p}{\sin\alpha}\\ +is\cos\alpha+iu\cos\alpha-i\cos^2\alpha\tan\alpha-is\frac{u}{\tan\alpha}\end{array}\right\} \tag{25}$$

where

$$\frac{i\left[\left(s+u-\tan\alpha\cos\alpha\right)^2+p^2\right]}{2\tan\alpha}=\frac{i\left(p^2+s^2+2su-2s\cos\alpha\tan\alpha+u^2-2u\cos\alpha\tan\alpha+\cos^2\alpha\tan^2\alpha\right)}{2\tan\alpha} \tag{26}$$

and

$$\frac{i\left(s+u-\tan\alpha\cos\alpha\right)p}{\sin\alpha}=ip\frac{s}{\sin\alpha}+ip\frac{u}{\sin\alpha}-ip \tag{27}$$

In that case, from equation (24), the following can be obtained:

$$\left\langle p\middle|F\middle|s+u-\frac{1}{2ab}\right\rangle\exp\left\{-ia\left[2su-\frac{u}{ab}-\frac{s}{ab}+\frac{1}{2a^2b^2}\right]\right\}\frac{c}{\sqrt{2\pi}}$$
$$=\left(\frac{c}{\sqrt{2\pi}}\right)^2\exp\left\{\begin{array}{l}\frac{i\left[\left(s+u-\tan\alpha\cos\alpha\right)^2+p^2\right]}{2\tan\alpha}-\frac{i\left(s+u-\tan\alpha\cos\alpha\right)p}{\sin\alpha}\\ +is\cos\alpha+iu\cos\alpha-i\cos^2\alpha\tan\alpha-is\frac{u}{\tan\alpha}\end{array}\right\}$$

$$=\frac{c}{\sqrt{2\pi}}\exp\left[\frac{i\left(p^2+s^2\right)}{2\tan\alpha}-\frac{ips}{\sin\alpha}\right]\frac{c}{\sqrt{2\pi}}\exp\left[\frac{i\left(p^2+u^2\right)}{2\tan\alpha}-\frac{ipu}{\sin\alpha}\right]$$

$$\exp\left[ip-\frac{ip^2}{2\tan\alpha}-\frac{1}{2}i\sin\alpha\cos\alpha\right]$$

$$=\langle p|F|s\rangle\langle p|F|u\rangle\exp i\left(p-ap^2-\frac{1}{2}\sin\alpha\cos\alpha\right) \tag{28}$$

$H_\alpha(p)$ can be rewritten as follows:

$$H_\alpha(p)$$

$$=\frac{c}{\sqrt{2\pi}}\iint dsdu\left\langle p\middle|F\middle|s+u-\frac{1}{2ab}\right\rangle\langle u|f\rangle\langle s|g\rangle\exp\left\{-ia\left[2su-\frac{u}{ab}-\frac{s}{ab}+\frac{1}{2a^2b^2}\right]\right\}$$

$$=\iint dsdu\langle p|F|u\rangle\langle u|f\rangle\langle p|F|s\rangle\langle s|g\rangle\exp\left(ip-\frac{1}{2}i\frac{p^2}{\tan\alpha}-\frac{1}{2}i\sin\alpha\cos\alpha\right)$$

$$=\langle p|F|f\rangle\langle p|F|g\rangle\exp\left(ip-\frac{1}{2}i\frac{p^2}{\tan\alpha}-\frac{1}{2}i\sin\alpha\cos\alpha\right)$$

$$=\Im_\alpha(p)\wp_\alpha(p)\exp\left[i\left(p-ap^2-\frac{1}{2}\sin\alpha\cos\alpha\right)\right] \tag{29}$$

where

$$\psi(x)=\exp i\left(p-ap^2-\frac{1}{2}\sin\alpha\cos\alpha\right) \tag{30}$$

and the following can be obtained:

$$\Im_\alpha[f \,\Box\, g](x)=\psi(x)\Im_\alpha[f](x)\Im_\alpha[g](x) \tag{31}$$

Definition 2 can be obtained using the same method as in Definition 1.

4. Conclusion

Based on quantum mechanical representation and operator theory, the two new convolutions of fractional Fourier transform proposed by Anh PK and other researchers have been proven using Dirac notation. The new convolutions can be sampled and reduced, which have potential use in signal processing.

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The authors declare no conflict of interest.

Author contributions

C.L. conceived the idea of the study. Y.C. performed the experiments, analyzed the data, and wrote the paper. N.H. and N.J. sorted the data.

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