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# Dissipativity of Multistep Runge–Kutta Methods for Nonlinear Neutral Delay-Integro-Differential Equations with Constrained Grid

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**Abstract:** This paper is concerned with the numerical dissipativity of multistep Runge-Kutta methods for nonlinear neutral delay-integro-differential equations. We investigate the dissipativity properties of -algebraically stable multistep Runge-Kutta methods with constrained grid. The finite-dimensional and infinite-dimensional dissipativity results of -algebraically stable multistep Runge-Kutta methods are obtained.

**Keywords:** Dissipativity; -algebraically stability; Nonlinear neutral delay- integro-differential equation; Multistep Runge-Kutta methods

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## 1 Introduction

Many dynamical systems in physics and engineering are characterized by the property of possessing a bounded absorbing set which all trajectories enter in a finite time and thereafter remain inside[1-4]. They are modeled by dissipative dynamical systems. In the study of dissipative systems it is often the asymptotic behavior of the system that is of interest, and so it is important to analyze whether or not numerical methods inherit the dissipativity of the dynamical systems when considering the applicability of numerical methods for these systems.

Humphries and Stuart[3, 4] first studied the dissipativity of Runge–Kutta methods for initial value problems (IVPs) of ordinary differential equations (ODEs) in1994, and proved that an algebraically stable, irreducible method can inherit the dissipativity of finite-dimesional systems. Later, many results on the dissipativity of numerical methods for ODEs have already been found[5-7]. For the delay differential equations (DDEs) with constant delay, Huang[8] gave a sufficient condition for the dissipativity of theoretical solution, and investigated the dissipativity of ($k$, $l$) -algebraically stable Runge–Kutta methods. Huang and Chen [9] and Huang[10], subsequently, obtained some results about the dissipativity of linear $\theta$-methods and G($c$, $p$, 0) -algebraically stable one-leg methods, respectively. In addition, Huang [11] further discussed the dissipativity of multistep Runge-Kutta methods, and proved that an algebraically stable, irreducible multistep Runge-Kutta methods with linear interpolation procedure is finite-dimensional dissipative. In 2004, Tian [12] studied the dissipativity of DDEs with a bounded variable lag and the dissipativity of $\theta$ -method. Moreover, Wen[13] discussed the dissipativity of Volterra functional differential equations, and further investigated the dissipativity of DDEs with piecewise delays and a class of linear multistep methods. In recent years, a number of works on the dissipativity of numerical methods have been carried out. Gan[14-16] studied the dissipativity of numerical methods for nonlinear integro differential equations(IDEs), nonlinear delay-integro-differential equations(DIDEs) and nonlinear pantograph equations, respectively. As to nonlinear Volterra delay-integro-differential Equations, it was

shown that for $\theta \in [1/2, 1]$, any linear $\theta$-method and one-leg method can inherit the dissipativity property, which was obtained by Gan[15]. In addition, Cheng and Huang [19], Wen et al[20] and Wang et al[21] considered the dissipativity for nonlinear neutral delay differential equations(NDDEs). Wu and Gan [22] consider the dissipativity for a class of nonlinear neutral delay integro differential equations (NDIDEs). So far we have not seen in literature more dissipativity results for nonlinear NDIDEs.

This paper pursues this, and further investigates the dissipativity of multistep Runge-Kutta methods for nonlinear NDIDEs. The motivations are as follows. Multistep Runge-Kutta methods are a wider class of methods which has as special cases not only one-leg methods, linear multistep methods, and Runge-Kutta methods, but also a wide range of hybrid methods. In particular, there exist algebraically stable multistep Runge-Kutta methods with only real eigenvalues such that they not only possess very good stability, but also can be performed in parallel.

## 2 The description of the problem and numerical Methods

Let $H$ be a real or complex, finite dimensional or infinite-dimensional Hilbert space with the inner product $\langle \cdot,\cdot \rangle$ and the corresponding induced norm $\|\cdot\|$, and the matrix norm is subordinated to the vector norm. $X$ be a dense continuously imbedded subspace of $H$. Consider the following initial value problems (IVPs) of nonlinear NDIDEs:

$$\begin{cases} \dfrac{d}{dt}[y(t)-Ny(t-\tau)] = f(y(t), y(t-\tau), \int_{t-\tau}^{t} g(t,\xi,y(\xi))d\xi), t \geq 0, \\ y(t) = \varphi(t), -\tau \leq t \leq 0. \end{cases} \tag{2.1}$$

where is a given constant delay, $N \in X \times X$ stands for a constant matrix with $\|N\| < 1$, $\varphi : [-\tau, 0] \to X$ is a continuous function, $f : X \times X \times X \to H$ is a locally Lipschitz continuous function, $g : [0,+\infty) \times [-\tau,+\infty) \times X \to X$ is a continuous function, $f$ and satisfy the following conditions:

$$\mathrm{Re}\langle u - Nv, f(u,v,w)\rangle \leq \beta_0 + \beta_1\|u\|^2 + \beta_2\|v\|^2 + \beta_3\|w\|^2, \quad u,v,w \in X, \tag{2.2}$$

and

$$\|g(t,s,u)\| \leq \eta\|u\|, \ t \in [0,+\infty), \ s \in [-\tau,+\infty), \ u \in X \tag{2.3}$$

where $\beta_0$, $\beta_1$, $\beta_2$, $\beta_3$ and $\eta$ are real constants.

Throughout this paper, we assume that the problem (2.1) has unique exact solution $y(t)$. For the study of solvability, we refer the reader to [2].

**Reark2.1.** When $N$=0, the problem (2.1) degenerates into an IVP of DIDEs. When the right-hand side function of the problem (2.1) does not possess the integral term, the problem (2.1) degenerates into an IVP of NDDEs. When $N$=0 and the right-hand side function of the problem (2.1) does not possess the integral term, the problem (2.1) degenerates into an IVP of DDEs. In the above various cases, the number of papers dealing with different aspects of their numerical integration now amounts to several hundreds.

**Proposition 2.2**[15] . Condition (2.2) implies that $\beta_0 \geqslant 0, \beta_2 \geqslant 0$ and $\beta_3 \geqslant 0$.

Next, let us consider the adaptation of $s$-stage multistep Runge-Kutta methods for solving problem (2.1) based on the formula

$$\begin{cases} Y_i^{(n)} - N\bar{Y}_i^{(n)} = \sum_{j=1}^{r} a_{ij}(y_{n+j-1} - N\bar{y}_{n+j-1}) + h\sum_{j=1}^{s} b_{ij} f(Y_j^{(n)}, \bar{Y}_j^{(n)}, \bar{G}_j^{(n)}), i = 1,2,\cdots,s. \\ y_{n+r} - N\bar{y}_{n+r} = \sum_{j=1}^{r} \theta_j (y_{n+j-1} - N\bar{y}_{n+j-1}) + h\sum_{j=1}^{s} \gamma_j f(Y_j^{(n)}, \bar{Y}_j^{(n)}, \bar{G}_j^{(n)}). \end{cases} \tag{2.4}$$

where $h$>0 is the fixed stepsize, the parameters $a_{ij}$, $b_{ij}$, $\theta_j$ and $\gamma_j$ are real constants, $Y_i^{(n)}$ and $y_n$ are approximation to $y(t_n+c_ih)$ and $y(t_n)$, respectively, and $t_n=nh$. The argument $\bar{Y}_i^{(n)}, \bar{G}_i^{(n)}$, and $\bar{y}_n$ denotes an approximation to $y(t_n + c_ih - \tau)$, $\int_{t_n+c_ih-\tau}^{t_n+c_ih} g(t_n + c_ih, \xi, y(\xi))d\xi$ and $y(t_n - \tau)$, those are obtained by a specific interpolation procedure using values $Y_i^{(k)}$ and $y_{k+r-1}$ $(k \leq n)$. The initial values $y_n = \varphi(t_n), \bar{y}_n = \varphi(t_n - \tau)$ for $t_n \leq 0$, and $Y_i^{(n)} = \varphi(t_n + c_ih)$, $\bar{Y}_i^{(n)} = \varphi(t_n + c_ih - \tau)$ for $t_n + c_ih \leq 0$. Following the referee's suggestion, we assume that $0 \leq c_i \leq 1$, $i = 1,2,\cdots,s$.

As to the computation of the delay terms $\bar{Y}_i^{(n)}$ and integral terms $\bar{G}_i^{(n)}$, $i = 1,2,\cdots,s$, we use the constrained stepsize $h$ satisfying $hm=t$ with a positive integer $m$. Let

$$\bar{Y}_i^{(n)} = Y_i^{(n-m)}, i = 1,2,\cdots,s, \tag{2.5a}$$

$$\bar{y}_n = y_{n-m}. \tag{2.5b}$$

and the compound quadrature (CQ) formula for the integral terms:

$$\bar{G}_i^{(n)} = h\sum_{q=0}^{m} \nu_q g(t_i^{(n)}, t_i^{(n-q)}, Y_i^{(n-q)}), \ i = 1,2,\cdots,s. \tag{2.5c}$$

where $t_i^{(n)} = t_n + c_ih$, $i = 1,2,\cdots,s$.

The quadrature formula (2.5c) can be derived from a uniform repeated rule [19]. For our stability analysis

we need the rule to satisfy the following condition:

$$h\sqrt{(m+1)\sum_{q=0}^{m}\left|v_q\right|^2}<v, \tag{2.6}$$

with $hm=t$ and a positive constant $v$.

**Remark 2.3.** We consider the procedure (2.5) here because, in the case that the order of the method is more than 2, there will be no order reduction if the corresponding quadrature rule is used. But it must be noticed that the stepsize $h$ is limited by $mh=t$ .

The used values $Y_i^{(n)}$ and $y_n$ with $n<-m<0$ are assumed to be 0. Here, we do not discuss other details.

It is well known that multistep Runge-Kutta methods are a subclass of a general linear methods. Let

$$C_{11}=[b_{ij}]\in R^{s\times s},\quad C_{12}=[a_{ij}]\in R^{s\times r}, \tag{2.7a}$$

$$C_{21}=\begin{bmatrix}0&\cdots&0\\ \vdots& &\vdots\\ 0&\cdots&0\\ \gamma_1&\cdots&\gamma_s\end{bmatrix}\in R^{r\times s},$$

$$C_{22}=\begin{bmatrix}0&1&0&\cdots&0\\ 0&0&1&\cdots&0\\ \vdots&\vdots&\ddots&\ddots&\vdots\\ 0&0&\cdots&0&1\\ \theta_1&\theta_2&\cdots&\theta_{r-1}&\theta_r\end{bmatrix}\in R^{r\times r} \tag{2.7b}$$

For any given $k\times l$ real matrix $Q=[q_{ij}]$, we define the corresponding linear operator $Q: X^l\to X^k$,

$QU=V=(v_1,v_2,\cdots,v_k)\in X^k$ ,

$U=(u_1,u_2,\cdots,u_l)\in X^l, u_j\in X$,

with

$$v_i=\sum_{j=1}^{l}q_{ij}u_j,\ i=1,2,\cdots,k.$$

Then, method (2.3) can be rewritten in the form of general linear method

$$\begin{cases}G^{(n)}=hC_{11}F(Y^{(n)},\overline{Y}^{(n)},\overline{G}^{(n)})+C_{12}g^{(n-1)},\\ g^{(n)}=hC_{21}F(Y^{(n)},\overline{Y}^{(n)},\overline{G}^{(n)})+C_{22}g^{(n-1)}.\end{cases} \tag{2.8}$$

with the following notational conventions:

$Y^{(n)}=(Y_1^{(n)},Y_2^{(n)},\cdots,Y_s^{(n)})^{\mathrm{T}}$,

$\overline{Y}^{(n)}=(\overline{Y}_1^{(n)},\overline{Y}_2^{(n)},\cdots,\overline{Y}_s^{(n)})^{\mathrm{T}}$,

$\overline{G}^{(n)}=(\overline{G}_1^{(n)},\overline{G}_2^{(n)},\cdots,\overline{G}_s^{(n)})^{\mathrm{T}}$

$G^{(n)}=(Y_1^{(n)}-N\overline{Y}_1^{(n)},Y_2^{(n)}-N\overline{Y}_2^{(n)},\cdots,Y_s^{(n)}-N\overline{Y}_s^{(n)})^{\mathrm{T}}$,

$g^{(n)}=(y_{n+1}-N\overline{y}_{n+1},y_{n+2}-N\overline{y}_{n+2},\cdots,y_{n+r}-N\overline{y}_{n+r})^{\mathrm{T}}$,

$F(Y^{(n)},\overline{Y}^{(n)},\overline{G}^{(n)})=(f(Y_1^{(n)},\overline{Y}_1^{(n)},\overline{G}_1^{(n)}),$

$f(Y_2^{(n)},\overline{Y}_2^{(n)},\overline{G}_2^{(n)}),\cdots,f(Y_s^{(n)},\overline{Y}_s^{(n)},\overline{G}_s^{(n)}))^{\mathrm{T}}$.

We introduce the following notations for brevity, for any real symmetric $p\times p$ matrix $Q=[q_{ij}]$, and $Q\geq 0(>0)$ means that $Q$ is nonnegative definite (positive definite). For any $Q\geq 0$, define a pseudo inner product on $H^{\mathrm{p}}$ by

$$\langle Y,Z\rangle_Q=\sum_{i,j=1}^{p}q_{ij}\langle Y_i,Z_j\rangle, Y=(Y_1,Y_2,\cdots,Y_p)\in H^p,$$

$Z=(Z_1,Z_2,\cdots,Z_p)\in H^p$,

and the corresponding pseudo norm on $H^{\mathrm{p}}$ by

$$\|Y\|_Q=\langle Y,Y\rangle_Q^{1/2}.$$

Especially $\|\cdot\|$ is the simplicity for $\|\cdot\|_Q$ when $Q$ is identity matrix.

**Definition2.4.** Let $k$, $l$ be real constants. A multistep Runge-Kutta method (2.4) is said to be($k$, $l$) -algebraically stable if there exists a real symmetric $r\times r$ matrix $G>0$ and a diagonal matrix $D=diag(d_1, d_2, \cdots, d_s)$ such that $M=[M_{ij}]\geq 0$ , where

$$M=\begin{bmatrix}kG-C_{22}^{\mathrm{T}}GC_{22}-2lC_{12}^{\mathrm{T}}DC_{12} & C_{12}^{\mathrm{T}}D-C_{22}^{\mathrm{T}}GC_{21}-2lC_{12}^{\mathrm{T}}DC_{11}\\ DC_{12}-C_{21}^{\mathrm{T}}GC_{22}-2lC_{11}^{\mathrm{T}}DC_{12} & C_{11}^{\mathrm{T}}D+DC_{11}-C_{21}^{\mathrm{T}}GC_{21}-2lC_{11}^{\mathrm{T}}DC_{11}\end{bmatrix} \tag{2.9}$$

As an important special case, a(1,0)-algebraically stable method is called algebraically stable for short.

**Definition2.5.** Let $l$ be a real constant, and $H$ be a finite-dimensional (or infinite-dimensional) space. A multistep Runge-Kutta method (2.4) with an interpolation procedure and integral terms are said to be finite-dimensionally (or infinite-dimensionally) $D(l)$ dissipative if, when the method is applied to problem (2.1) in $H$ withstepsize $h$ satisfying

$d(\beta_1h+\beta_2h+h\beta_3\eta^2v^2)<ld_{\min}(1-\|N\|)^2$ ,

(or $d(\beta_1h+\beta_2h+h\beta_3\eta^2v^2)<ld(1-\|N\|)^2$)

and constraint $t=mh$, where $d_{\min}=\min_{1\leq j\leq s}d_j$ and $d=\sum_{j=1}^{s}d_j$, there exists a constant $C$ such that, for any initial values, there exists an $n_0$ , dependent only on initial values, such that

$\|y_n\|\leq C$ , $n\geq n_0$ ,

holds. As an important special case, a $D(0)$ -dissipative method is called $D$-dissipative for short.

$GD(l)$ -and $GD$-dissipativity are defined by dropping restriction $t=mh$.

**Definition 2.6**[11]. A multistep Runge-Kutta method (2.4) is said to be stage-reducible if, for some nonempty index set $T\subset\{1,2,\cdots,s\}$,

$\gamma_j=0,\ for\ \ j\in T$,

$b_{ij} = 0,\ for\ \ j \in T, i \notin T$ .

Otherwise , it is said to be stage-irreducible.

**Definition 2.7.** A multistep Runge-Kutta method (2.4) is said to be step-reducible if polynomials $\{\sigma_i(x)\}_{i=0}^{s}$ have common divisor where

$$\sigma_0(x) = x^r - N\bar{x}^r - \sum_{j=1}^{r}\theta_j(x^{j-1} - N\bar{x}^{j-1}),$$

$$\sigma_i(x) = \sum_{j=1}^{r} a_{ij}(x^{j-1} - N\bar{x}^{j-1}), i = 1,2,\cdots,s.$$

Otherwise , it is said to be step-irreducible.

**Definition 2.8**[11]. A multistep Runge-Kutta method (2.4) is said to be reducible if it is stage-reducible or step-reducible.

## 3 Finite-dimensional numerical dissipativity

In this section, we focus on the dissipativity analysis of (*k*, *l*)-algebraically stable multistep Runge-Kutta methods with respect to nonlinear NDIDEs in finite-dimensional spaces. We always assume that $H=X=C^N$.

**Lemma 3.1**[11]. Suppose $\{\xi_i(x)\}_{i=1}^{r}$ are a basis of polynomials for $P^{r-1}$, the space of polynomials of degree strictly less than *r* and *E* is the translation operator: $Ey_n = y_{n+1}$. Then there is always a unique solution $y_n, y_{n+1}, \ldots, y_{n+r-1}$ to the system of equations $\xi_i(E)y_n = \Delta_i$, $\Delta_i \in C^N$, $i = 1,2,\cdots,r.$

and there exists a constant $\chi$ , independent of $\Delta_i$ , such that

$$\max_{0\le i\le r-1}\|y_{n+i}\| \le \chi \max_{0\le i\le r-1}\|\Delta_i\|.$$

**Lemma 3.2**[11]. Suppose that a multistep Runge-Kutta method (2.4) is step-irreducible. Then, there exist real constants $v_i, i = 1,2,\cdots,s$, such that $\sigma_0(x)$ and $\sum_{i=1}^{s} v_i\sigma_i(x)$ have no common divisor.

Now we state and prove the main results.

**Theorem 3.3.** Assume that a step-irreducible multistep Runge-Kutta method (2.4) is (*k*, *l*) -algebraically stable, $D>0$, $l>0$ and $k \le 1$, the problem (2.1) satisfies (2.2) and (2.3) with $d(\beta_1 h + \beta_2 h + h\beta_3\eta^2 v^2) < ld_{\min}(1-\|N\|)^2$. Then the method (2.4) with (2.5) is finite-dimensionally $D(l)$-dissipative.

Proof. From (2.5), using Cauchy-Schwarz inequality we can obtain

$$\|\bar{Y}^{(j)}\|^2 = \|Y^{(j-m)}\|^2 \tag{3.1a}$$

$$\|\bar{y}_n\| = \|y_{n-m}\| \tag{3.1b}$$

$$\|\bar{G}_i^{(n)}\|^2 = \left\|h\sum_{q=0}^{m} v_q g(t_i^{(n)}, t_i^{(n-q)}, Y_i^{(n-q)})\right\|^2 \le h^2\eta^2\sum_{q=0}^{m}|v_q|^2\sum_{q=0}^{m}\|Y_i^{(n-q)}\|^2$$

$$\le \frac{\eta^2 v^2}{m+1}\sum_{q=0}^{m}\|Y_i^{(n-q)}\|^2$$

Therefore,

$$\|\bar{G}^{(n)}\|^2 \le \frac{\eta^2 v^2}{m+1}\sum_{q=0}^{m}\|Y^{(n-q)}\|^2 \tag{3.1c}$$

As in [17] and [15], by means of (*k*, *l*)-algebraically stability of the method, we can easily obtain that

$$\|g^{(n)}\|_G^2 - k\|g^{(n-1)}\|_G^2 - 2\,\mathrm{Re}\langle G^{(n)}, hF(Y^{(n)}, \bar{Y}^{(n)}, \bar{G}^{(n)})\rangle_D + 2l\|G^{(n)}\|_D^2 \tag{3.2}$$

$$= \Big\langle C_{21}hF(Y^{(n)}, \bar{Y}^{(n)}, \bar{G}^{(n)}) + C_{22}g^{(n-1)},$$

$$G(C_{21}hF(Y^{(n)}, \bar{Y}^{(n)}, \bar{G}^{(n)}) + C_{22}g^{(n-1)})\Big\rangle$$

$$+ \left\langle g^{(n-1)}, -kGg^{(n-1)})\right\rangle$$

$$+ 2\,\mathrm{Re}\Big\langle C_{11}hF(Y^{(n)}, \bar{Y}^{(n)}, \bar{G}^{(n)})$$

$$+ C_{12}g^{(n-1)}, -DhF(Y^{(n)}, \bar{Y}^{(n)}, \bar{G}^{(n)})\Big\rangle$$

$$+ \Big\langle C_{11}hF(Y^{(n)}, \bar{Y}^{(n)}, \bar{G}^{(n)}) + C_{12}g^{(n-1)},$$

$$2lD(C_{11}hF(Y^{(n)}, \bar{Y}^{(n)}, \bar{G}^{(n)}) + C_{12}g^{(n-1)})\Big\rangle$$

$$= -\Big\langle\Big\langle g^{(n-1)}, hF(Y^{(n)}, \bar{Y}^{(n)}, \bar{G}^{(n)})\Big\rangle,$$

$$M\Big\langle g^{(n-1)}, hF(Y^{(n)}, \bar{Y}^{(n)}, \bar{G}^{(n)})\Big\rangle\Big\rangle \le 0$$

Considering (2.2), (2.3) and $k \le 1$, we have

$$\|g^{(n)}\|_G^2 \le \|g^{(n-1)}\|_G^2 + 2hd\beta_0 + 2\beta_1 h\|Y^{(n)}\|_D^2$$

$$+ 2h\beta_2\|\bar{Y}^{(n)}\|_D^2 + 2h\beta_3\|\bar{G}^{(n)}\|_D^2 - 2l\|G^{(n)}\|_D^2,$$

$$\le \|g^{(n-1)}\|_G^2 + 2hd\beta_0 + 2\beta_1 dh\|Y^{(n)}\|^2$$

$$+ 2hd\beta_2\|\bar{Y}^{(n)}\|^2 + 2hd\beta_3\|\bar{G}^{(n)}\|^2$$

$$- 2ld_{\min}(\|Y^{(n)}\|^2 + \|N\|^2\|\bar{Y}^{(n)}\|^2 - 2\|N\|\langle Y^{(n)}, \bar{Y}^{(n)}\rangle)$$

Using (3.1) and Cauchy-Schwarz inequality, we have

$$\|g^{(n)}\|_G^2 \le \|g^{(n-1)}\|_G^2 + 2hd\beta_0 + 2\beta_1 dh\|Y^{(n)}\|^2$$

$$+ 2hd\beta_2\|\bar{Y}^{(n)}\|^2 + 2hd\beta_3\frac{\eta^2 v^2}{m+1}\sum_{q=0}^{m}\|Y^{(n-q)}\|^2$$

$$- 2ld_{\min}(\|Y^{(n)}\|^2 + \|N\|^2\|\bar{Y}^{(n)}\|^2) + 2ld_{\min}\|N\|(\|Y^{(n)}\|^2 + \|\bar{Y}^{(n)}\|^2)$$

$$\le [2\beta_1 dh - 2ld_{\min}(1-\|N\|)]\|Y^{(n)}\|^2 + [2hd\beta_2 + 2ld_{\min}\|N\|(1-\|N\|)]\|\bar{Y}^{(n)}\|^2$$

$$+ \|g^{(n-1)}\|_G^2 + 2hd\beta_0 + 2hd\beta_3\frac{\eta^2 v^2}{m+1}\sum_{q=0}^{m}\|Y^{(n-q)}\|^2 \tag{3.3}$$

Where

$$d_{\min} = \min_{1\le j\le s} d_j,\quad d = \sum_{j=1}^{s} d_j. \tag{3.4}$$

By induction, we can easily obtain

$$\left\|g^{(n)}\right\|_G^2 \le \left\|g^{(-1)}\right\|_G^2 + 2(n+1)hd\beta_0$$
$$+\left[2\beta_1 dh - 2ld_{\min}(1-\|N\|)\right]\sum_{j=0}^{n}\left\|Y^{(j)}\right\|^2$$
$$+\left[2hd\beta_2 + 2ld_{\min}\|N\|(1-\|N\|)\right]\sum_{j=0}^{n}\left\|\overline{Y}^{(j)}\right\|^2$$
$$+ 2hd\beta_3\frac{\eta^2 v^2}{m+1}\sum_{j=0}^{n}\sum_{q=0}^{m}\left\|Y^{(j-q)}\right\|^2. \quad (3.5)$$

When using (2.5) and (3.1) on substitution into (3.5) gives

$$\left\|g^{(n)}\right\|_G^2 \le \left\|g^{(-1)}\right\|_G^2 + 2(n+1)hd\beta_0 + 2hd\beta_3\frac{\eta^2 v^2}{m+1}$$
$$\left[(m+1)\sum_{j=0}^{m}\left\|Y^{(j)}\right\|^2 + \frac{m(m+1)}{2}\max_{-m\le i\le -1}\left\|Y^{(i)}\right\|^2\right]$$
$$+\left[2d\beta_1 h - 2ld_{\min}(1-\|N\|)\right]\sum_{j=0}^{n}\left\|Y^{(j)}\right\|^2$$
$$+\left[2hd\beta_2 + 2ld_{\min}\|N\|(1-\|N\|)\right]\sum_{j=0}^{n}\left\|Y^{(j-m)}\right\|^2$$
$$\le \left\|g^{(-1)}\right\|_G^2 + 2(n+1)hd\beta_0 + 2\left[d\beta_1 h + d\beta_2 h\right.$$
$$\left. - ld_{\min}(1-\|N\|)^2 + hd\beta_3\eta^2 v^2\right]\sum_{j=0}^{n}\left\|Y^{(j)}\right\|^2$$
$$+\left[2d\beta_2\tau + 2ld_{\min}\|N\|(1-\|N\|)m + d\beta_3\eta^2 v^2\tau\right]\max_{-m\le i\le -1}\left\|Y^{(i)}\right\|^2 \quad (3.6)$$

Let $\lambda_1$ denote the maximum eigenvalue of the matrix $G$,

$$a_1 = \max_{0\le i\le r-1}\|y_i\|^2,\ a_2 = \max_{-m\le i\le -1}\left\|Y^{(i)}\right\|^2,$$
$$R_1 = \max(a_1, a_2),$$
$$\mu = ld_{\min}(1-\|N\|)^2 - d(\beta_1 h + \beta_2 h + h\beta_3\eta^2 v^2).$$

Then, we have $\mu > 0$ and

$$\left\|g^{(n)}\right\|_G^2 + 2\mu\sum_{j=0}^{n}\left\|Y^{(j)}\right\|_D^2 \le \left[2d\beta_2\tau + 2ld_{\min}\|N\|(1-\|N\|)m\right.$$
$$\left. + d\beta_3\eta^2 v^2\tau\right]\max_{-m\le i\le -1}\left\|Y^{(i)}\right\|^2 + \left\|g^{(-1)}\right\|_G^2 + 2(n+1)hd\beta_0$$
$$\le r\lambda_1(1+\|N\|)a_1 + \left[2d\beta_2\tau + 2ld_{\min}\|N\|(1-\|N\|)m\right.$$
$$\left. + d\beta_3\eta^2 v^2\tau\right]a_2 + 2(n+1)hd\beta_0$$
$$\le \left[r\lambda_1(1+\|N\|) + 2d\beta_2\tau + 2ld_{\min}\|N\|(1-\|N\|)m\right.$$
$$\left. + d\beta_3\eta^2 v^2\tau\right]R_1 + 2(n+1)hd\beta_0. \quad (3.7)$$

When $\beta_0 = 0$, it follows from (3.7) and $\mu > 0$ that

$$\lim_{n\to\infty}\left\|Y^{(n)}\right\| = 0,$$

which shows that for any $\varepsilon > 0$, there exists $n_0(R_1, \varepsilon) > 0$, such that

$$\left\|Y_j^{(n)}\right\| \le \varepsilon,\ \left\|\overline{Y}_j^{(n)}\right\| \le \varepsilon\ \ j = 1,2,\cdots,s,\ n \ge n_0 \quad (3.8)$$

Hence, (2.4) implies that

$$\left\|\sigma_i(E)(y_n - N\overline{y}_n)\right\| = \left\|\sum_{j=1}^{r} a_{ij}(y_{n+j-1} - N\overline{y}_{n+j-1})\right\| \le hL\sum_{j=1}^{s}|b_{ij}| + \varepsilon,$$
$$i = 1,2,\cdots,s,\ n \ge n_0 \quad (3.9a)$$

$$\left\|\sigma_0(E)(y_n - N\overline{y}_n)\right\| = \left\|y_{n+r} - N\overline{y}_{n+r}\right.$$
$$\left. - \sum_{j=1}^{r}\theta_j(y_{n+j-1} - N\overline{y}_{n+j-1})\right\| \le hL\sum_{j=1}^{s}|\gamma_j|, \quad (3.9b)$$

where

$$L = \sup_{\|u\|\le\varepsilon,\|v\|\le\varepsilon,\|w\|\le\varepsilon}\|f(u,v,w)\|,\ u,v,w \in X.$$

From Lemma 3.2 it follows that there exist real constants $v_i$, $i = 1,2,\cdots,s$, such that $\sigma_0(x)$ and $\sum_{i=1}^{s} v_i\sigma_i(x)$ have no common divisor. Therefore,

$$\left\|\sum_{i=1}^{s} v_i\sigma_i(E)(y_n - N\overline{y}_n)\right\| \le \sum_{i=1}^{s}|v_i|\left[hL\sum_{j=1}^{s}|b_{ij}| + \varepsilon\right],\ n \ge n_0,$$

which further gives

$$\left\|\left[\sigma_0(E) - \sum_{i=1}^{s} v_i\sigma_i(E)\right](y_n - N\overline{y}_n)\right\| \le hL$$
$$\sum_{j=1}^{s}|\gamma_j| + \sum_{i=1}^{s}|v_i|\left[hL\sum_{j=1}^{s}|b_{ij}| + \varepsilon\right],\ n \ge n_0 \quad (3.10)$$

Since $\sigma_0(x)$ and $\sigma_0(x) - \sum_{i=1}^{s} v_i\sigma_i(x)$ are coprime, and both are of degree r. Hence,

$$\left\{x^i\sigma_0(x), x^i[\sigma_0(x) - \sum_{i=1}^{s} v_i\sigma_i(x)]:\right.$$
$$\left. i = 0,1,\cdots,r-1.\right\},$$

form a basis for $P^{2r-1}$. Considering (3.9), (3.10) and Lemma 3.1, we have $\left\|y_n - N\overline{y}_n\right\| \le \chi\left[hL\sum_{j=1}^{s}|\gamma_j| + \sum_{i=1}^{s}|v_i|\left(hL\sum_{j=1}^{s}|b_{ij}| + \varepsilon\right)\right]$, for $n \ge n_0$ and $\beta_0 = 0$.

Therefore,

$$\|y_n\| \le \|N\|\|\overline{y}_n\| + \chi\left[hL\sum_{j=1}^{s}|\gamma_j| + \sum_{i=1}^{s}|v_i|\left(hL\sum_{j=1}^{s}|b_{ij}| + \varepsilon\right)\right]$$

$$\le \|N\|\|y_{n-m}\| + \chi\left[hL\sum_{j=1}^{s}|\gamma_j| + \sum_{i=1}^{s}|v_i|\left(hL\sum_{j=1}^{s}|b_{ij}| + \varepsilon\right)\right]$$

$$\le \frac{\chi\left[hL\sum_{j=1}^{s}|\gamma_j| + \sum_{i=1}^{s}|v_i|\left(hL\sum_{j=1}^{s}|b_{ij}| + \varepsilon\right)\right]}{1-\|N\|} + \max_{-\tau\le\xi\le 0}\|\varphi(\xi)\| \quad (3.11)$$

where $n \ge n_0', n_0' = m + n_0$.

When $\beta_0 > 0$, let us take $n = 2(m+r)q - 1$,

$$q = \left\lfloor \frac{\left[r\lambda_1(1+\|N\|) + 2d\beta_2\tau + 2ld_{\min}\|N\|(1-\|N\|)m + d\beta_3\eta^2v^2\tau\right]R_1}{4(m+r)hd\beta_0} \right\rfloor + 1,$$

where the notation $\lfloor x \rfloor$ means the maximum integer no greater than $x$, then

$$\left[r\lambda_1(1+\|N\|) + 2d\beta_2\tau + 2ld_{\min}\|N\|(1-\|N\|)m + d\beta_3\eta^2v^2\tau\right]R_1 \le 4(m+r)qhd\beta_0.$$

It follows from (3.7) that

$$\mu\sum_{j=0}^{2(m+r)q-1}\left\|Y^{(i)}\right\|^2 \le 4(m+r)qhd\beta_0,$$

which gives

$$\sum_{i=0}^{2q-1}\sum_{j=(m+r)i}^{(m+r)(i+1)-1}\left\|Y^{(i)}\right\|^2 \le 4(m+r)\frac{qhd\beta_0}{\mu}.$$

Hence, there exists an integer such that $c \in [q, 2q-1]$.

$$\sum_{j=(m+r)c}^{(m+r)(c+1)-1}\left\|Y^{(i)}\right\|^2 \le 4(m+r)\frac{hd\beta_0}{\mu} \quad (3.12)$$

Let $p = (m+r)c + m$, then for all ,we have $j \in [p-m, p+r-1]$,

$$\left\|Y^{(i)}\right\|^2 \le a_2' \quad (3.13)$$

where

$$a_2' = 4(m+r)\frac{hd\beta_0}{\mu}.$$

Therefore, by (2.4) and (2.5), for all $n \in [p, p+r-1]$,

$$\|\sigma_i(E)(y_n - N\bar{y}_n)\| = \left\|\sum_{j=1}^{r}a_{ij}(y_{n+j-1} - N\bar{y}_{n+j-1})\right\|$$

$$\le hL_1\sum_{j=1}^{s}|b_{ij}| + \sqrt{\frac{a_2'}{d_i}}, i = 1,2,\cdots,s. \quad (3.14a)$$

$$\|\sigma_0(E)(y_n - N\bar{y}_n)\| = \left\|y_{n+r} - N\bar{y}_{n+r} - \sum_{j=1}^{r}\theta_j(y_{n+j-1} - N\bar{y}_{n+j-1})\right\| \le hL_1\sum_{j=1}^{s}|\gamma_j|, \quad (3.14b)$$

where

$L_1 = \sup_{\|y\|\le w,\|z\|\le w,\|\omega\|\le w}\|f(y,z,\omega)\|, y,z,\omega \in X$,

where $w = \sqrt{a_2'/d_{\min}}$.

Therefore,

$$\left\|\left[\sigma_0(E) - \sum_{i=1}^{s}v_i\sigma_i(E)\right](y_n - N\bar{y}_n)\right\| \le hL_1\sum_{j=1}^{s}|\gamma_j| + \sum_{i=1}^{s}|v_i|\left[hL_1\sum_{j=1}^{s}|b_{ij}| + \sqrt{\frac{a_2'}{d_i}}\right], \quad (3.15)$$

with $n \in [p, p+r-1]$.

Considering (3.14), (3.15), lemma 3.1 and lemma 3.2, similar to (3.11), we have

$$\|y_n - N\bar{y}_n\|^2 \le a_1', n \in [p, p+r-1], \quad (3.16)$$

where

$$a_1' = \chi^2\left[hL_1\sum_{j=1}^{s}|\gamma_j| + \sum_{i=1}^{s}|v_i|\left(hL_1\sum_{j=1}^{s}|b_{ij}| + \sqrt{\frac{a_2'}{d_i}}\right)\right]^2.$$

Let

$R_2 = \max(a_1', a_2')$.

A repetition of the above analysis implies that there exists a ,

$p' \in [p + (m+r)q' + m, p + (2q'-1)(m+r) + m]$

$$q' = \left\lfloor \frac{\left[r\lambda_1(1+\|N\|) + 2d\beta_2\tau + 2ld_{\min}\|N\|(1-\|N\|)m + d\beta_3\eta^2v^2\tau\right]R_2}{4(m+r)hd\beta_0} \right\rfloor + 1,$$

such that

$$\left\|Y^{(n)}\right\|^2 \le a_2', \quad n \in [p'-m, p'+r-1], \quad (3.17)$$

$$\|y_n - N\bar{y}_n\|^2 \le a_1', n \in [p', p'+r-1]. \quad (3.18)$$

Similar to (3.3), (3.5) and (3.7), for $n \in [p, p']$, we can obtain

$$\left\|g^{(n-1)}\right\|_G^2 \le \left\|g^{(p-1)}\right\|_G^2 + 2(n-p)hd\beta_0 + \left[2d\beta_2\tau + 2ld_{\min}\|N\|(1-\|N\|)m + d\beta_3\eta^2v^2\tau\right]\max_{p-m\le i\le p-1}\left\|Y^{(i)}\right\|^2 \quad (3.19)$$

$$\le 2\left(\left[2d\beta_2\tau + 2ld_{\min}\|N\|(1-\|N\|)m + d\beta_3\eta^2v^2\tau\right] + r\lambda_1(1+\|N\|)\right)R_2 + 2(2m+r)hd\beta_0$$

Similar to (3.11), we can obtain

$$\|y_n\| \le \|N\|\|\bar{y}_n\| +$$

$$\sqrt{2\left(\left[2d\beta_2\tau + 2ld_{\min}\|N\|(1-\|N\|)m + d\beta_3\eta^2v^2\tau\right] + r\lambda_1(1+\|N\|)\right)R_2 + 2(2m+r)hd\beta_0}$$

$$\le \|N\|\|y_{n-m}\| +$$

$$\sqrt{2\left(\left[2d\beta_2\tau + 2ld_{\min}\|N\|(1-\|N\|)m + d\beta_3\eta^2v^2\tau\right] + r\lambda_1(1+\|N\|)\right)R_2 + 2(2m+r)hd\beta_0}$$

Hence, by induction, we have

$$\|y_n\| \le$$

$$\frac{\sqrt{2\left(\left[2d\beta_2\tau + 2ld_{\min}\|N\|(1-\|N\|)m + d\beta_3\eta^2v^2\tau\right] + r\lambda_1(1+\|N\|)\right)R_2 + 2(2m+r)hd\beta_0}}{1-\|N\|}$$

$$+ \max_{-\tau\le\xi\le 0}\|\varphi(\xi)\| \qquad (3.20)$$

for

$$n \ge \frac{\left(\left[2d\beta_2\tau + 2ld_{\min}\|N\|(1-\|N\|)m + d\beta_3\eta^2v^2\tau\right] + r\lambda_1(1+\|N\|)\right)R_2}{2hd\beta_0} + 2m + r$$

and $\beta_0 > 0$.

A combination of (3.11) and (3.20) shows that the method is finite-dimensionally $D(l)$-dissipative.

**Theorem 3.4.** Assume that a step-irreducible multistep Runge-Kutta method (2.4) is ($k$, $l$) -algebraically stable, $D>0$, $l<0$ and $k \le 1$, the problem (2.1) satisfies (2.2) and (2.3) with $d(\beta_1 h + \beta_2 h + h\beta_3\eta^2v^2) < ld_{\min}(1-\|N\|)^2$. Then the method (2.4) with (2.5) is finite-dimensionally $D(l)$-dissipative.

Proof. In the proof of theorem 3.3, change all $d_{\min}$ into $d$, we can get the proof of theorem 3.4.

**Theorem 3.5.** Assume that a method (2.4) is irreducible and algebraically stable, the problem (2.1) satisfies (2.2) and (2.3) with $\beta_1 h + \beta_2 h + \beta_3 h < 0$. Then, the method (2.4) with (2.5) is finite-dimensionally $D$-dissipative.

Proof. As in [18], we can prove that, if a stage-irreducible method (2.4) is algebraically stable for the matrices $G$ and $D$, then $D>0$, therefore, use the proof of theorem 3.3 for $k=1$, $l=0$, we prove this theorem.

## 4 Infinite- dimensional numerical dissipativity

In this section, we further discuss the dissipativity of ($k$, $l$)-algebraically stable multistep Runge-Kutta methods in infinite-dimensional spaces. Here we assume that $H$ is an infinite-dimensional complex Hilbert space instead of $H = X = C^N$.

**Theorem 4.1.** Assume that a step-irreducible multistep Runge-Kutta method (2.4) is ($k$, $l$)-algebraically stable, $D>0$, $l>0$ and $k<1$, the problem (2.1) satisfies (2.2), (2.3) with $d(\beta_1 h + \beta_2 h + h\beta_3\eta^2v^2) < ld_{\min}(1-\|N\|)^2$. Then the method (2.4) with (2.5) is infinite-dimensionally $D(l)$-dissipative.

Proof. Let

$$\mu = d(\beta_1 h + \beta_2 h + h\beta_3\eta^2v^2) - ld_{\min}(1-\|N\|)^2, \qquad (4.1)$$

when $d(\beta_1 h + \beta_2 h + h\beta_3\eta^2v^2) < ld_{\min}(1-\|N\|)^2$, we have $\mu < 0$. Using (3.3), we deduce that

$$\left\|g^{(n)}\right\|_G^2 \le k\left\|g^{(n-1)}\right\|_G^2 + 2hd\beta_0 + \left[2d\beta_1 h - 2ld_{\min}(1-\|N\|)\right]\left\|Y^{(n)}\right\|^2$$
$$+ \left[2hd\beta_2 + 2ld_{\min}\|N\|(1-\|N\|)\right]\left\|\bar{Y}^{(n)}\right\|^2$$
$$+ 2hd\beta_3\frac{\eta^2v^2}{m+1}\sum_{q=0}^{m}\left\|Y^{(n-q)}\right\|^2 \qquad (4.2)$$

Iterating (4.2) and considering (2.6) and (3.1), we have

$$\left\|g^{(n)}\right\|_G^2 \le k^{n+1}\left\|g^{(-1)}\right\|_G^2 + 2\sum_{i=0}^{n}k^{n-i}$$
$$\left[hd\beta_0 + \left[d\beta_1 h - ld_{\min}(1-\|N\|)\right]\left\|Y^{(i)}\right\|^2\right]$$
$$+ 2\sum_{i=0}^{n}k^{n-i}\left[dh\beta_2 + ld_{\min}\|N\|(1-\|N\|)\right]\left\|\bar{Y}^{(i)}\right\|^2$$
$$+ 2hd\beta_3\frac{\eta^2v^2}{m+1}\sum_{i=0}^{n}k^{n-i}\sum_{q=0}^{m}\left\|Y^{(i-q)}\right\|^2$$
$$\le k^{n+1}\left\|g^{(-1)}\right\|_G^2 + 2\sum_{i=0}^{n}k^{n-i}$$
$$\left[hd\beta_0 + \left[d\beta_1 h - ld_{\min}(1-\|N\|)\right]\left\|Y^{(i)}\right\|^2\right]$$
$$+ 2\sum_{i=0}^{n}k^{n-i}\left[dh\beta_2 + ld_{\min}\|N\|(1-\|N\|)\right]\left\|Y^{(i-m)}\right\|^2$$
$$+ 2hd\beta_3\frac{\eta^2v^2}{m+1}\sum_{i=0}^{n}k^{n-i}\sum_{q=0}^{m}\left\|Y^{(i-q)}\right\|^2$$
$$\le k^{n+1}\left\|g^{(-1)}\right\|_G^2 + 2\sum_{i=0}^{n}k^{n-i}\left(hd\beta_0 + \mu\left\|Y^{(i)}\right\|^2\right)$$

$$+\left[2d\beta_2\tau + 2ld_{\min}\|N\|(1-\|N\|)m + 2d\beta_3\eta^2v^2\tau\right]k^{n-m}\max_{-m\le i\le -1}\left\|Y^{(i)}\right\|^2$$
$$\le k^{n+1}\left\|g^{(-1)}\right\|_G^2 + \frac{2hd\beta_0}{1-k} + 2\sum_{i=0}^{n}k^{n-i}\mu\left\|Y^{(i)}\right\|^2$$

$$+\left[2d\beta_2\tau + 2ld_{\min}\|N\|(1-\|N\|)m + 2d\beta_3\eta^2v^2\tau\right]k^{n-m}\max_{-m\le i\le -1}\left\|Y^{(i)}\right\|^2$$
$$\le k^{n+1}\left\|g^{(-1)}\right\|_G^2 + \frac{2hd\beta_0}{1-k} +$$
$$\left[2d\beta_2\tau + 2ld_{\min}\|N\|(1-\|N\|)m + 2d\beta_3\eta^2v^2\tau\right]k^{n-m}\max_{-m\le i\le -1}\left\|Y^{(i)}\right\|^2 \qquad (4.3)$$

where we have used that $\mu < 0$.

Thus, for any $\varepsilon > 0$ there exists $n_0>0$, such that

$$\left\|g^{(n)}\right\|_G \le \sqrt{\frac{2hd\beta_0}{1-k} + \varepsilon}, \; n \ge n_0. \qquad (4.4)$$

Similar to (3.11), we can obtain

$$\|y_n\| \le \|N\|\|\bar{y}_n\| + \sqrt{\frac{2hd\beta_0}{1-k}+\varepsilon}, \quad n \ge n_0,$$
$$\le \|N\|\|y_{n-m}\| + \sqrt{\frac{2hd\beta_0}{1-k}+\varepsilon}$$
$$\le \frac{\sqrt{\frac{2hd\beta_0}{1-k}+\varepsilon}}{1-\|N\|} + \max_{-\tau\le\xi\le 0}|\varphi(\xi)|, \quad n \ge n_0'.$$

with $n_0' = m + n_0$, which shows the method (2.4) with (2.5) is infinite-dimensionally $D(l)$-dissipative.

**Theorem 4.2.** Assume that a step-irreducible multistep Runge-Kutta method (2.4) is ($k$, $l$)-algebraically stable, $D>0$, $l<0$ and $k<1$, the problem (2.1) satisfies (2.2) ,(2.3) with $d(\beta_1 h + \beta_2 h + h\beta_3\eta^2 v^2) < ld_{\min}(1-\|N\|)^2$. Then the method (2.4) with (2.5) is infinite-dimensionally $D(l)$-dissipative.

Proof. In the proof of theorem 4.1, change all $d_{\min}$ into $d$, we can get the proof of theorem 4.2.

**Theorem 4.3.** Assume that there exist nonnegative constants $k_1$, $k_2$, $k_1<k_2$, such that the stage-irreducible method (2.4) is $((1-k_1 l)/(1-k_2 l), l)$-algebraically stable for every $l<0$. Then, the method (2.4) with (2.5) is infinite-dimensionally $D$-dissipative.

## 5 Comparison with existing results

(1) When $N=0$, the problem (2.1) degenerates into an IVP of DIDEs

$$\begin{cases} y'(t) = f(y(t), y(t-\tau), \int_{t-\tau}^{t} g(t,\xi, y(\xi))d\xi), t \ge 0, \\ y(t) = \varphi(t), -\tau \le t \le 0. \end{cases} \tag{5.1}$$

The conditions (2.2) and (2.3) degenerate into

$$\mathrm{Re}\langle u, f(u,v,w)\rangle \le \beta_0 + \beta_1\|u\|^2 + \beta_2\|v\|^2 + \beta_3\|w\|^2, \quad u,v,w \in X, \tag{5.2}$$

and

$$\|g(t,s,u)\| \le \eta\|u\|, \ t \in [0,+\infty), \ s \in [-\tau,+\infty), \ u \in X \tag{5.3}$$

Gan [15] studies the dissipativity of $\theta$-methods for DIDEs (5.1), Qi et al. [23] study the dissipativity of multistep Runge-Kutta methods for nonlinear VDIDEs. So far we have not seen in literature other numerical dissipativity results for nonlinear DIDEs. But Theorems 3.3, Theorems 3.4, Theorems 4.1 and Theorems 4.2 in this paper can be applied to this class of problem directly, and we can obtain the following Corollaries.

**Corollary 5.1.** Assume that a step-irreducible multistep Runge-Kutta method (2.4) is ($k$, $l$)-algebraically stable, $D>0$, $l>0$ and $k\le 1$, the problem (5.1) satisfies (5.2) , (5.3) with $d(\beta_1 h + \beta_2 h + h\beta_3\eta^2 v^2) < ld_{\min}$. Then the method (2.4) with (2.5) for DIDEs is finite-dimensionally $D(l)$-dissipative.

**Corollary 5.2.** Assume that a step-irreducible multistep Runge-Kutta method (2.4) is ($k$, $l$)-algebraically stable, $D>0$, $l>0$ and $k<1$, the problem (5.1) satisfies (5.2) , (5.3) with $d(\beta_1 h + \beta_2 h + h\beta_3\eta^2 v^2) < ld_{\min}$ . Then the method (2.4) with (2.5) for DIDEs is infinite-dimensionally $D(l)$-dissipative.

(2) When the right-hand side function of the problem (2.1) does not possess the integral term, the problem (2.1) degenerates into an IVP of NDDEs

$$\begin{cases} \dfrac{d}{dt}[y(t) - Ny(t-\tau)] = f(y(t), y(t-\tau)), t \ge 0, \\ y(t) = \varphi(t), -\tau \le t \le 0. \end{cases} \tag{5.4}$$

and the fourth term of the right side of condition (2.2) vanishes and thus (2.2) degenerates into

$$\mathrm{Re}\langle u - Nv, f(u,v)\rangle \le \beta_0 + \beta_1\|u\|^2 + \beta_2\|v\|^2, \quad u,v \in X \tag{5.5}$$

Wen [20] studies the dissipativity of $\theta$-methods for nonlinear NDDEs (5.4) ,Wang[21] studies the dissipativity of Runge-Kutta methods for NDDEs with piecewise constant delay. So far we have not seen in literature other numerical dissipativity results for nonlinear NDDEs. Theorems 3.3, Theorems 3.4, Theorems 4.1 and Theorems 4.2 in this paper can be applied to this class of problem directly, and we can obtain the following Corollaries.

**Corollary 5.3.** Assume that a step-irreducible multistep Runge-Kutta method (2.4) is ($k$, $l$) -algebraically stable, $D>0$, $l>0$ and $k \le 1$, the problem (5.4) satisfies (5.5) with $d(\beta_1 h + \beta_2 h) < ld_{\min}$ . Then the method (2.4) with (2.5) for NDDEs is finite-dimensionally $D(l)$-dissipative.

**Corollary 5.4.** Assume that a step-irreducible multistep Runge-Kutta method (2.4) is ($k$, $l$) -algebraically stable, $D>0$, $l>0$ and $k<1$, the problem (5.4) satisfies (5.5) with $d(\beta_1 h + \beta_2 h) < ld_{\min}$ . Then the method (2.4) with (2.5) for NDDEs is infinite-dimensionally $D(l)$-dissipative.

(3) When $N=0$ and the right-hand side function of the problem (2.1) does not possess the integral term,

the problem (2.1) degenerates into an IVP of DDEs. Therefore, the results of Theorems 3.3, Theorems 3.4, Theorems 4.1 and Theorems 4.2 In this paper partially cover the numerical dissipativity of multistep Runge-Kutta for DDEs which is given by Huang in [11].

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