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A Study on the Largest Size of Self-Conjugate Simultaneous Core Partitions

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Abstract: For a positive integer s, a partition is said to be s-core if its hook length set avoids hook length s. The theory of s-core partitions has intriguing applications in representation theory, number theory, and combinatorics. Analogous to the work of Xiong on the largest size of an $(s, s+1, \ldots, s+k)$-core partition, we evaluate the largest size of a self-conjugate $(s, s+1, \ldots, s+k)$-core partition for given positive integers s and k. This generalizes the result on the largest size of a self-conjugate $(s, s+1, \ldots, s+k)$-core partition, which is obtained by Baek, Nam, and Yu by employing Johnson's bijection.

Keywords: Core partition; Self-conjugate

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1. Introduction

Recall that a partition is a finite weakly decreasing sequence of positive integers $\lambda=(\lambda_1,\lambda_2,\ldots\lambda_m)$. The size of λ, denoted by $|\lambda|$, is defined to be. For a positive integer s, a partition is said to be s-core if its hook length set avoids hook length s. See **Figure 1** for an illustration of a 7-core partition λ. Recently, various results on the number, the average size, and the largest size of partitions whose hook sets avoid multiple lengths, also known as simultaneous core partitions, have been derived since Anderson's work [1]. In particular, Olsson and Stanton [2] and Tripathi [3] independently confirmed Aukerman, Kane, and Sze's conjecture [4] concerning the largest size of an (s, t)-core partition for coprime positive integers s and t. Xiong [5] further derived the largest size of an -core partition for positive integers $(s, s+1, \ldots, s+k)$ and k.

The study of self-conjugate simultaneous core partitions was initiated by Ford, Mai, and Sze [6]. Analogous to the work of Xiong [5], we derive the following result, which generalizes the result of Baek, Nam, and Yu [7] concerning the largest size of a self-conjugate $(s, s+1, s+2)$ -core partition.

Theorem 1.1 Let s and k be positive integers. Then the largest size of a self-conjugate $(s, s+1, \dots, s+k)$ -core partitions are given by

$$\max\{p_1^2(q_1+1)+2sp_1\binom{q_1+1}{2}-\frac{1}{4}(2sk+k^2)\binom{2q_1+2}{3},$$
$$(2tp_2+p_2^2)(q_2+1)+2(p_2s-kt)\binom{q_2+1}{2}-\frac{1}{4}(2sk+k^2)\binom{2q_2+2}{3}\},$$

where $p_1=\lfloor \frac{s}{2} \rfloor$,$q_1=\lfloor \frac{\lfloor \frac{s}{2}\rfloor -1}{k} \rfloor$,$p_2=\lfloor \frac{s-k}{2} \rfloor$,$q_2=\lfloor \frac{\lfloor \frac{s-k}{2}\rfloor -1}{k} \rfloor$ and $t=\lceil \frac{s+k}{2} \rceil$. Moreover, there are at most two self-conjugate $(s, s+1, \dots, s+k)$-core partitions having the largest size.

The case $k=2$ in Theorem 1.1 implies the following result on the largest size of a self-conjugate $(s, s+1, s+2)$-core partition.

Corollary 1.2 (See [7], Theorem 3.3) There is a unique self-conjugate $(s, s+1, s+2)$-core partition λ of maximum size, where

$$\lambda|=\begin{cases} \frac{n(2n+1)(4n^2+2n+1)}{3} & \textit{if } s=4n, \\ \frac{n^2(8n^2-6n+1)}{3} & \textit{if } s=4n-1, \\ \frac{n(2n-1)(4n^2-2n+1)}{3} & \textit{if } s=4n-2, \\ \frac{(n-1)(2n-1)(4n^2-5n+3)}{3} & \textit{if } s=4n-3. \end{cases}$$

The case $k=1$ in Theorem 1.1 reduces the following result on the largest size of a self-conjugate $(s, s+1)$-core partition.

Corollary 1.3 There is a unique self-conjugate $(s, s+1)$-core partition λ of maximum size, where

$$|\lambda|=\begin{cases} \frac{n(n+1)(2n-1)(2n+1)}{6} & \textit{if } s=2n, \\ \frac{n(n+1)(4n^2+8n+3)}{6} & \textit{if } s=2n+1. \end{cases}$$

2. The proof

In this section, we shall evaluate the largest size of a self-conjugate $(s, s+1, \dots, s+k)$ -core partition for given positive integers s and k.

First we recall some definitions and notations. For a partition λ, denote by $MD(\lambda)$ the set of its main diagonal hook lengths in its Young diagram. For instance, **Figure 1** shows the Young diagram and the hook length set of a self-conjugate partition $\lambda=(6, 4,2, 2,1,1)$ with $MD(\lambda)=\{11, 5\}$.

11	8	5	4	2	1
8	5	2	1		
5	2				
4	1				
2					
1					

Figure 1. The Young diagram and the hook length set of λ= (6,4, 2, 2,1,1)

It is easily seen that $MD(\lambda)$ consists of distinct odd integers if and only if λ is self-conjugate. Moreover, we have

$$\lambda| = \sum_{x\in MD(\lambda)} x \qquad (2.1)$$

Given a poset $(P,<)$, for two elements x and y, we say that y covers x if $x<y$ and there exists no element $z\in P$ with $x<y<z$. An element x of a poset P is said to be a minimal element of P if there exists no element y in P such that $y<x$.

Definition 2.1 Set $L(s,k)=\bigcup_{i\geq 0} L_i$, *and* $R(s,k)=\bigcup_{i\geq 0} R_i$, where

$$L_i=\{2j-1+2si \mid ik+1\leq j\leq \lfloor \frac{s}{2} \rfloor \}$$

and

$$R_i=\{2j-1+2si \mid ik+\lceil \frac{s+k}{2} \rceil +1\leq j\leq s\}$$

Define

$$P(s,k)=L(s,k)\cup R(s,k)$$

with the following cover relation: if $x,y\in P(s,k)$, then y covers x if and only if $y=x+2s+2t$ for some $0\leq t\leq k$.

In the following lemma, Yan, Yu, and Zhou [9] presented a characterization of the posets $L(s,k)$ and $R(s,k)$.

Lemma 2.2 (See [8]) For any positive odd integer x with $x\geq 2S$, we have $x\in L(s,k)$(resp. $x\in R(s,k)$) if and only if x-$2k$-$2t\in L(s,k)$ (resp. x-$2k\in R(s,k)$) for all $0\leq t\leq k$.

We may identify a finite poset P with its Hasse diagram, which is defined to be a graph in which the vertices are the elements of P, and there is an edge connecting x and y if and only if y covers x. See **Figures 2** and **3** for an illustration of the Hasse diagrams of $P(18; 3)$ and $P(18; 4)$. A subset I of P is said to be an order ideal of P if for any $y\in I$, the elements covered by y are also contained in I. Denote by $J(P)$ the set of all order ideals of P. An order ideal I of $P(s,k)$ is said to be proper if there do not exist two elements x_1,x_2 with the property that $x_1+x_2=2s+2t$ for all $0\leq t\leq k$. Denote by $J^*(P(s,k))$ the set of all proper order ideals of .

Relying on Ford, Mai and Sze's characterization [6] of the set $MD(\lambda)$ of the self- conjugate s-core partition λ, Yan, Yu, and Zhou [8] further established a correspondence between self-conjugate (s, s + 1, …, s + k) -core partitions and the proper order ideals of the poset $P(s,k)$.

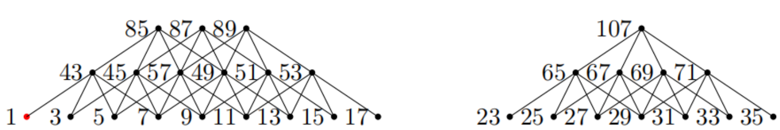

Figure 2. An example of the Hasse diagram of the poset $P(18,4)$

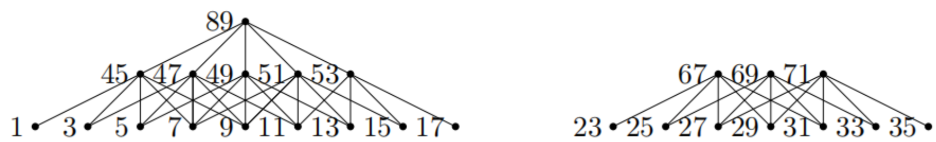

Figure 3. An example of the Hasse diagram of the poset $P(18,4)$

Theorem 2.3 (See [8]) The partition λ is a self-conjugate $(s, s+1, \ldots, s+k)$-core partition if and only if $MD(\lambda)$ is a proper order ideal of the poset $P(s,k)$.

The following two lemmas give a complete characterization of the set $MD(\lambda)$ of a self-conjugate $(s, s+1, \ldots, s+k)$-core partition of the largest size.

Lemma 2.4 If λ is a self-conjugate $(s, s+1, \ldots, s+k)$-core partition of the maximum size, then we have either $MD(\lambda) \in J(L(s;k))$ or $MD(\lambda) \in J(R(s;k))$.

Proof. If not, suppose that x is the smallest element of $MD(\lambda)$ which is a minimal element and is contained in $L(s,k)$, and y is the largest element of $MD(\lambda)$ which is a minimal element and is contained in $R(s,k)$. Now we proceed to construct a new proper order ideal I of $P(s,k)$. We have two cases.

If $x<2s-y$, then let I be the set obtained from $MD(\lambda)$ by replacing each element $p\in MD(\lambda)$ by the element $p+y+1$ if $p\geq q$ for some $q<2s-y$ and $q\in L_0$. We proceed to show that I is a proper order ideal of $P(s,k)$. By Lemma 2.2, it suffices to show that for any $p\in L_0$ and $p<2s-y$, we have $p+y+1\in R_0$. Since $1\leq x\leq p\leq 2s-y-2$, we have $y+2\leq p+y+1\leq 2s-1$. Hence, we conclude that I is a proper order ideal of $P(s,k)$. For example, let $MD(\lambda)=\{1,3,5,7,43,23\}\in J^*(P(18,3))$. Clearly, we have $=1,y=23$ and $x<2s-y$. By the above construction of I, we get a new proper order ideal $I=\{23,25,27,29,31\}$ of $P(18,3)$.

If $x>2s-y$, then let z be the element of $MD(\lambda)$ such that $z<2s+2k-x$. We choose z to be the smallest such element. Let I be the set obtained from $MD(\lambda)$ by replacing each element $p\in MD(\lambda)$ by the element $p+2s+2k+1$-z if $p\geq q$ for some $q\geq z$ and $q\in R_0$, and adjoining the elements $1,3,\ldots,x-2$. We claim that I is a proper order ideal of $P(s,k)$. By Lemma 2.2, it suffices to show that for any $p\in R_0$ and $q\geq z$, we have $p+2s+2k+1-z\in L(s,k)$. Since $2s+2k-x\leq p\leq 2s-1$, we have $1\leq p+2s+2k+1-z-(2s+2t)=p-z+2k-2t+1\leq x-2-2t$ for all $0\leq t\leq k$. Recall that $x-2-2t\in I$ for all $0\leq t\leq k$. By Lemma 2.2, for any $p\in R_0$ and $p\geq z$, we have $p+2s+2k+1$-z$\in L(s,k)$. For example, let $MD(\lambda)=\{29,31,33,35,17\}\in J^*(P(18,4))$. It is easy to check that $x=17,y=35,z=29$ and $x>2s-y$. By the above construction of I, we get a new proper order ideal $I=\{1,3,5,7,9,11,13,15,17,45,47,49,51\}$ of $P(18,4)$.

One can easily check that we have obtained a new proper order ideal I of $P(s,k)$ which has a larger sum of elements in both cases. By Relation (2.1), the size of the self-conjugate $(s, s+1, \ldots, s+k)$-core partition corresponding to I is larger than that of λ. This yields a contradiction with the assumption that λ is a self-conjugate $(s, s+1, \ldots, s+k)$-core partition of the largest size, completing the proof.

Lemma 2.5 Let λ be a self-conjugate $(s, s+1, \ldots, s+k)$-core partition. If λ is of maximum size, then we have either $MD(\lambda)=L(s,k)$ or $MD(\lambda)=R(s,k)$.

Proof. Theorem 2.3 together with Lemma 2.2 tells us that the self-conjugate partitions corresponding to $L(s,k)$ and $R(s,k)$ are $(s, s+1, \ldots, s+k)$-core partitions. From Lemma 2.4, it follows that we have either $MD(\lambda)\subseteq L(s,k)$ or $MD(\lambda)\subseteq R(s,k)$. Then the assertion follows immediately from Relation 2.1, completing the proof.

Proof of Theorem 1.1 Let $\alpha_{s,k}$ and $\beta_{s,k}$ denote the self-conjugate partition such that $MD(\alpha_{s,k})=L(s,k)$ and

$MD(\alpha_{s,k})=R(s,k)$. By Relation (2.1), we have

$$\begin{aligned}|\alpha_{s,k}| &= \sum_{x\in L(s,k)} x\\ &=\sum_{i=0}^{q_1}\left((s+k)(p_1-ki)2i+(p_1-ki)^2\right)\\ &=\sum_{i=0}^{q_1}\left(p_1^2+2sp_1 i-(2sk+k^2)i^2\right)\\ &=p_1^2(q_1+1)+2sp_1\binom{q_1+1}{2}-\frac{1}{4}(2sk+k^2)\binom{2q_1+2}{3},\end{aligned}$$

and

$$\begin{aligned}\beta_{s,k}| &= \sum_{x\in R(s,k)} x\\ &=\sum_{i=0}^{q_2}\left((2t+2i(s+k))(p_2-ki)+(p_2-ki)^2\right)\\ &=\sum_{i=0}^{q_2}\left(p_2(2t+p_2)+2i(p_2s-kt)-i^2(2sk+k^2)\right)\\ &=(2tp_2+p_2^2)(q_2+1)+2(p_2s-kt)\binom{q_2+1}{2}-\frac{1}{4}(2sk+k^2)\binom{2q_2+2}{3}.\end{aligned}$$

Hence, the assertion follows immediately from Lemma 2.5, completing the proof.

Proof of Corollary 1.2. Here we only deal with the cases when s=4n nd s=4n-1. The other two cases can be verified by similar arguments, and the details of the discussion are omitted. By Theorem 1.1, we have $p_1=2n, q_1=n-1, p_2=2n-1, q_2=n-1, t=2n+1$.

Hence, we have

$$\begin{aligned}\alpha_{4n,2}| &=p_1^2(q_1+1)+2sp_1\binom{q_1+1}{2}-\frac{1}{4}(2sk+k^2)\binom{2q_1+2}{3}\\ &=4n^3+8n^3(n-1)-\frac{2n(n-1)(2n-1)(4n+1)}{3}\\ &=\frac{2n(2n-1)(2n^2+3n+1)}{3},\end{aligned}$$

and

$$\begin{aligned}|\beta_{4n,2}| &=(2tp_2+p_2^2)(q_2+1)+2(p_2s-kt)\binom{q_2+1}{2}-\frac{1}{4}(2sk+k^2)\binom{2q_2+2}{3}\\ &=n(2n-1)(6n+1)+n(n-1)(8n^2-8n-2)-\frac{2n(n-1)(2n-1)(4n+1)}{3}\\ &=\frac{n(8n^3+8n^2+4n+1)}{3}\\ &=\frac{n(2n+1)(4n^2+2n+1)}{3}.\end{aligned}$$

Since $|\beta_{4n,2}|-|\alpha_{4n,2}|=2n^2+n>0$ for all $n>0$, the largest size of a self-conjugate $(4n,4n+1,4n+2)$-core partition is given by $\frac{n(2n+1)(4n^2+2n+1)}{3}$.

When s=4n-1, we have $p_1=2n-1, q_1=n-1, p_2=2n-2, q_2=n-2, t=2n+1$. Hence,

$$\begin{aligned}|\alpha_{4n-1,2}| &= p_1^2(q_1+1)+2sp_1\binom{q_1+1}{2}-\frac{1}{4}(2sk+k^2)\binom{2q_1+2}{3}\\ &= n^2(2n-1)(4n-3)-\frac{8n^2(n-1)(2n-1)}{3}\\ &= \frac{n^2(2n-1)(4n-1)}{3}.\end{aligned}$$

And

$$\begin{aligned}|\beta_{4n-1,2}| &= (2tp_2+p_2^2)(q_2+1)+2(p_2s-kt)\binom{q_2+1}{2}-\frac{1}{4}(2sk+k^2)\binom{2q_2+2}{3}\\ &= n(n-1)(8n^2-18n+16)-\frac{8n(n-1)(n-2)(2n-3)}{3}\\ &= \frac{2n^2(n-1)(4n+1)}{3}.\end{aligned}$$

ince $|\alpha_{4n-1,2}|-|\beta_{4n-1,2}|=n^2>0$ for all $n>0$, the largest size of a self-conjugate (4n-1,4n,4n+1)-core partition is given by $\frac{n^2(2n-1)(4n-1)}{3}$. This completes the proof.

Proof of Corollary 1.3 Here we only consider the case when $s=2n$. By similar arguments, one can verify the case when $s=2n+1$. It is apparent that we have $p_1=n, q_1=n-1, p_2=n-1, q_2=n-2, t=n+1$.. By Theorem 1.1, we have

$$\begin{aligned}|\alpha_{2n,1}| &= p_1^2(q_1+1)+2sp_1\binom{q_1+1}{2}-\frac{1}{4}(2sk+k^2)\binom{2q_1+2}{3}\\ &= n^3+2n^3(n-1)-\frac{n(n-1)(2n-1)(4n+1)}{6}\\ &= \frac{n(n+1)(2n-1)(2n+1)}{6},\end{aligned}$$

and

$$\begin{aligned}|\beta_{2n,1}| &= (2tp_2+p_2^2)(q_2+1)+2(p_2s-kt)\binom{q_2+1}{2}-\frac{1}{4}(2sk+k^2)\binom{2q_2+2}{3}\\ &= (n-1)(2n^3-4n^2+3n+1)-\frac{(n-1)(n-2)(2n-3)(4n+1)}{6}\\ &= \frac{n(n-1)(4n^2+2n+1)}{6}.\end{aligned}$$

By simple computation, we have $|\alpha_{2n,1}|-|\beta_{2n,1}|=n^3>0$ for all $n>0$. This implies that the largest size of self-conjugate (2n,2n+1)-core partition is $\frac{n(n+1)(2n-1)(2n+1)}{6}$, completing the proof.

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